# Mathematical Methods of Physics

## Physics 116A- Winter 2018

**Solution of the Final Examination, Total 100 Points**
March 20, 2018

1. Find the disk of convergence for the series

$$\sum_{n=0}^{\infty} \frac{(n!)^2 (z-\pi+i)^n}{(2n)!}.$$

...[15]

**§Solution** We can shift to $w = z - \pi + i$ to simplify the series to the form

$$\sum_{n=0}^{\infty} \frac{(n!)^2 w^n}{(2n)!} = \sum_n w^n a_n,$$

where $a_n = \frac{(n!)^2}{(2n)!}$. Applying the ratio test we can say that the series is convergent for

$$|w| < |w|_c = \lim_{n\to\infty} \frac{a_n}{a_{n+1}}.$$

We work out the limit

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \frac{(n!)^2(2n+2)!}{(2n)![(n+1)!]^2} = \frac{(2n+2)(2n+1)}{(n+1)^2} \sim 4.$$

Hence the series converges for

$$|z - \pi + i| < 4.$$

2. Consider the following two lines that intersect at point $I(0,1,2)$ written in parametric form:

$$\vec{r}_1 = (-4+2t,\ 1,\ t)$$
$$\vec{r}_2 = (12+4t,\ 4+t,\ -1-t)$$

a) Find an equation of a plane that contains both lines. ...[10]

b) Find the *shortest* distance from the plane to origin $O(0,0,0)$. ...[10]

**§Solution to both parts:**

We begin by identifying the direction of each line: $\vec{n}_1 = (2,0,1)$ and $\vec{n}_2 = (4,1,-1)$. Next we take the cross product of $\vec{n}_1$ and $\vec{n}_2$ to find the normal vector to the plane that contains both lines:

$$\vec{N} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 0 & 1 \\ 4 & 1 & -1 \end{vmatrix} = (-1,\ 6,\ 2)$$

Recall that every vector that lies in the plane satifies the condition $\vec{N}\cdot(\vec{r}-\vec{r}_0)=0$ where $\vec{r}=(x,y,z)$ and $\vec{r}_0=\overrightarrow{IO}=(0,1,2)$. Hence, an equation of the plane is

$$\vec{N}\cdot(\vec{r}-\vec{r}_0)=-x+6y+2z-10=0\ .$$

**For the second part:** *Find the shortest distance from the plane to origin $O(0,0,0)$.*

The shortest distance from the origin to the plane is given by $d=|\hat{\mathbf{N}}\cdot\vec{v}|$ where $\hat{\mathbf{N}}$ is the unit vector normal to the plane and $\vec{v}$ is a vector from an arbituary point on the plane, $R(x',y',z')$, to the point of interest (the origin). In the case that we choose the point of intersection, $I(0,1,2)$ as our arbitary point that lies on the plane, we have $\vec{v}=\overrightarrow{IO}=(0,1,2)$. The unit vector normal to the plane is $\hat{\mathbf{N}}=(-1,6,2)/\sqrt{41}$. Thus the distance from the origin to the plane is

$$d=|(-1,6,2)\cdot(0,1,2)/\sqrt{41}|=10/\sqrt{41}\ .$$

3. a) Evaluate

$$z=\left(\frac{1+2i}{1-2i}\right)^2$$

...[8]

§**Solution** We write the polar form

$$\left(\frac{1+2i}{1-2i}\right)=\frac{(1+2i)^2}{5}=\frac{-3+4i}{5}=\rho e^{i\phi}$$

where

$$\rho\cos\phi=-\frac{3}{5},\quad \text{and}\quad \rho\sin\phi=\frac{4}{5}$$

Thus

$$\rho=1,\ \ \tan(\phi)=-\frac{4}{3}.$$

Let us call this angle as $\phi_0$. Thus

$$z=e^{2i\phi_0}.$$

b) Express

$$\text{cotanh}(\log z)+\tanh(\log z)$$

in rectangular form. ...[7]

§**Solution** We first note that $\log(z)=2i\phi_0$. Also we note that

$$\tanh(i2\phi_0)=i\tan(2\phi_0)=i2\frac{\tan(\phi_0)}{1-\tan^2(\phi_0)}.$$

Now using $\tan(\phi) = -\frac{4}{3}$ we get

$$\tanh(i2\phi_0) = i2\frac{-4/3}{1-\frac{16}{9}} = i\frac{24}{7}.$$

Therefore

$$\coth(i2\phi_0) = -i\frac{7}{24},$$

and the answer follows

$$\text{cotanh}(\log z) + \tanh(\log z) = i\left(\frac{24}{7} - \frac{7}{24}\right) \sim i\ 3.1369$$

4. Consider the $2 \times 2$ matrix $A$ given as:

$$A = \begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}$$

(a) Find the two right eigenvectors and two left eigenvectors. . . . [10]

§**Solution** To find the right eigenvectors we take the operation of A on a general vector

$$(A - \lambda\mathbb{1}) \begin{bmatrix} x \\ y \end{bmatrix} = 0,$$

giving us two simultaneous equations

$$\begin{aligned} (1-\lambda)x - 2y &= 0 \\ (1+\lambda)y &= 0 \end{aligned}$$

The eigenvalues and (normalized) eigenvectors are

$$\lambda = 1,\ \psi_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix};$$
$$\lambda = -1,\ \psi_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

To find the left eigenvectors we operate to the left

$$\begin{bmatrix} x & y \end{bmatrix} (A - \lambda\mathbb{1}) = 0,$$

giving us two simultaneous equations

$$\begin{aligned} (1-\lambda)x &= 0 \\ 2x + (1+\lambda)y &= 0 \end{aligned}$$

with the same eigenvalues as before but different eigenvectors

$$\lambda = 1,\ \phi_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \end{bmatrix}.$$
$$\lambda = -1,\ \phi_2 = \begin{bmatrix} 0 & 1 \end{bmatrix};$$

(b) Combine the two right eigenvectors into a matrix $C$ and similarly the two left eigenvectors into a matrix $D$. Show that $D \propto C^{-1}$ ...[5]

**§Solution**

By combining the two right eigenvectors we find the matrix $C$ as

$$C = \begin{bmatrix} 1 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

and from the left eigenvectors we get

$$D = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 1 \end{bmatrix}$$

Hence we check that

$$D.C = \frac{1}{\sqrt{2}}\mathbb{1},$$

thus proving $C^{-1} = \sqrt{2}D$.

(c) From above show that

$$C^{-1}.A.C = A_D,$$

where $A_D$ is a diagonal matrix. ...[5]

**§Solution**

We can calculate as follows:

$$C^{-1}.A.C = \sqrt{2}D.A.C = \sqrt{2}D.\begin{bmatrix} 1 & -\frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} = \sqrt{2}\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 1 \end{bmatrix}.\begin{bmatrix} 1 & -\frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(d) Using the above information calculate explicitly the matrix

$$B = e^A$$

...[10]

{Hint: Recall that if we know the $C$ matrix we can find any function of $A$.}

**§Solution**

We can use the result of Part(c) to write

$$B = C^{-1}.e^{A_D}.C = C^{-1}.\begin{bmatrix} e & 0 \\ 0 & e^{-1} \end{bmatrix}.C$$

We can calculate the sandwich using $C$ as

$$B = \begin{bmatrix} e & \frac{1}{\sqrt{2}}(e - e^{-1}) \\ 0 & e^{-1} \end{bmatrix}$$

§**Extra comment on problem** 4.$(d)$

This problem has a somewhat strange feature that the answer depends upon the way we normalize the vectors. If the vectors in the matrix $C$ are not normalized to unity but rather we choose them as

$$C' = \begin{bmatrix} 1 & a \\ 0 & a \end{bmatrix}$$

then the normalization choice is recovered by putting $a = \frac{1}{\sqrt{2}}$. Another choice of $a$ we may make is $a = 1$ so that the vector is not normalized, and indeed this is usually OK too- although I would recommend that you should normalize whenever possible.

So the question is what happens with an arbitrary $a$. Here is the answer: the inverse is

$$C'^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & \frac{1}{a}, \end{bmatrix}$$

and we can check that proceeding as before

$$C'^{-1}.e^{A_D}.C' = \begin{bmatrix} e & a(e - e^{-1}) \\ 0 & e^{-1} \end{bmatrix}.$$

The scoring gives full credit to any choice made.

5. (a) Find the cofactor matrix of the matrix

$$M = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

...[8]

§**Solution**

The cofactor matrix is found as

$$C = \begin{bmatrix} -4 & 2 & -1 \\ 6 & -3 & -3 \\ -1 & -4 & 2 \end{bmatrix}$$

(b) Using the above find the inverse of the matrix M. ...[10]

§**Solution**

The transpose of the cofactor matrix is

$$C^T = \begin{bmatrix} -4 & 6 & -1 \\ 2 & -3 & -4 \\ -1 & -3 & 2 \end{bmatrix}$$

By taking the product we check easily

$$C^T.M = -9 \times \mathbb{1}$$

Hence

$$M^{-1} = -\frac{1}{9}C^T.$$

(c) What is the determinant of $M$?. . . . [2]
§**Solution** Clearly $DetM = -9$ since the $C^T.M = -9\mathbb{1}$.