Mathematical Methods of Physics 116A- Winter 2018

Physics 116A

Home Work # 1

Posted on Jan 11, 2018

Due in Class Jan 18, 2018

§Required Problems: Each problem has 5 points

E.g. MB 19.16 means problem #16 on page 19 in the book by M. Boas, 3rd Edition.

1. MB 3.3

$$\begin{aligned} 0.555555\ldots &= \frac{5}{10} + \frac{5}{10^2} + \frac{5}{10^3} + \cdots \\ &= \frac{5}{10}(1 + \frac{1}{10} + \frac{1}{10^2} + \cdots) \\ &= \frac{5}{10}\sum_{n=0}^{\infty}(1/10)^n \\ &= \frac{5}{10}\frac{1}{1-1/10} = \frac{5}{10}\frac{10}{9} \\ &= \boxed{5/9} \end{aligned} \tag{1}$$

2. MB 3.11

$$\begin{aligned} 0.678571428571428571\ldots &= 0.25 + 0.428571428571428571 \\ &= 0.25 + \frac{428571}{10^6} + \frac{428571}{(10^6)^2} + \frac{428571}{(10^6)^3} + \cdots \\ &= 0.25 + \frac{428571}{10^6}(1 + \frac{1}{10^6} + \frac{1}{(10^6)^2} + \cdots) \\ &= 0.25 + \frac{428571}{10^6}\sum_{n=0}^{\infty}(1/10^6)^n \\ &= \frac{1}{4} + \frac{428571}{10^6}\frac{1}{1-1/10^6} = \frac{1}{4} + \frac{428571}{999999} \\ &= \boxed{19/28} \end{aligned} \tag{2}$$

3. MB 5.6 Find the limit of the sequence.

$$\begin{aligned} \frac{n^n}{n!} &= \prod_{i=0}^{n-1}\frac{n}{n-i} \\ &= \prod_{i=0}^{n-1}\frac{1}{1-i/n} \end{aligned}$$

$$\lim_{n\to\infty}\prod_{i=0}^{n-1}\frac{1}{1-i/n}=\infty$$

4. MB 5.8 Find the limit of the sequence.

$$\begin{aligned}
\frac{(n!)^2}{(2n)!} &= \frac{(n!)}{(2n)!/n!} \\
&= \frac{n(n-1)(n-2)\cdots 3\cdot 2\cdot 1}{2n(2n-1)(2n-2)\cdots(n+3)(n+2)(n+1)} \\
&= \prod_{i=0}^{n-1}\left(\frac{n-i}{2n-i}\right) \\
&= \prod_{i=0}^{n-1}\left(\frac{1-i/n}{2-i/n}\right)
\end{aligned}$$

$$\lim_{n\to\infty}\prod_{i=0}^{n-1}\left(\frac{1-i/n}{2-i/n}\right)=0$$

5. MB 8.6 Find the formulas and the limiting values of the sequences, a_n, S_n and R_n:
The sequence formula a_n for the series is

$$a_n=\frac{1}{n(n+1)}\ ,$$

and the limiting value is

$$a=\lim_{n\to\infty}a_n\to 0\ .$$

The formula for the partial sum sequence S_n is

$$\begin{aligned}
S_n=\sum_{n=1}^{\infty}\frac{1}{n(n+1)} &= \sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right) \\
&= \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right) \\
&= \left(1-\frac{1}{n+1}\right),
\end{aligned}$$

and its limiting value is

$$S=\lim_{n\to\infty}S_n\to 1\ .$$

The remainder formula R_n is

$$R_n=S-S_n=\frac{1}{n+1}\ ,$$

and the limiting value is

$$R = \lim_{n\to\infty} R_n \to 0\ .$$

6. MB 8.7 Find the formulas and limiting values of the sequences, a_n, S_n and R_n:

$$\frac{2}{1\cdot 2} - \frac{5}{2\cdot 3} + \frac{7}{3\cdot 4} - \frac{9}{4\cdot 5} + \cdots$$

The sequence formula a_n for the series above is

$$\begin{aligned} a_n &= (-1)^{n+1}\frac{2n+1}{n(n+1)} \\ &= (-1)^{n+1}\left(\frac{2}{n+1} + \frac{1}{n(n+1)}\right) \\ &= (-1)^{n+1}\left(\frac{1}{n+1} + \frac{1}{n}\right)\ , \end{aligned}$$

and the limiting value is

$$a = \lim_{n\to\infty} (-1)^{n+1}\left(\frac{1}{n+1} + \frac{1}{n}\right) \to 0\ .$$

The sequence formula S_n is

$$\begin{aligned} S_n &= \{3/2, -5/3, 7/9, -9/20, \ldots\} \\ &= \{1+1/2, 1-1/3, 1+1/4, 1-1/5, \ldots\} \\ &= \left\{1 + \frac{(-1)^{n+1}}{(n+1)}\right\}_{n=1}^{\infty}\ , \end{aligned}$$

and the limiting value is

$$\begin{aligned} S &= \lim_{n\to\infty} S_n \\ &= \lim_{n\to\infty}\left(1 + \frac{(-1)^{n+1}}{(n+1)}\right) \to 1\ . \end{aligned}$$

The remainder formula R_n is

$$\begin{aligned} R_n &= S - S_n \\ &= \frac{(-1)^n}{n+1}\ , \end{aligned}$$

and the limiting value is

$$R = \lim_{n\to\infty} R_n \to 0\ .$$

7. MB 11.3 Use the comparison test to show that $\sum_{n=1}^{\infty} 1/n^2$ converges. Note that

$$\frac{1}{n(n-1)} > \frac{1}{n^2} \text{ for } n \geq 2$$

(since $n(n-1) < n^2$). Hence if $\sum_{n=2}^{\infty} 1/n(n-1)$ converges, then so does $\sum_{n=2}^{\infty} 1/n^2$. Next we decompose the sum into a partial fraction

$$\frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}.$$

The sum is a telescoping series

$$\begin{aligned}\sum_{n=2}^{N} \left(\frac{1}{n-1} - \frac{1}{n}\right) &= \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) \\ &+ \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots + \left(\cancel{\frac{1}{N-1}} - \frac{1}{N}\right) \\ &= 1 - \frac{1}{N},\end{aligned}$$

where $\lim_{N\to\infty}(1 - 1/N) = 1$, $\boxed{\text{so } \sum_{n=1}^{\infty} 1/n^2 \text{ also converges.}}$

8. MB 13.11 Use the integral test to determine the convergence or divergence of the sum:

$$\sum_{n=1}^{\infty} \frac{1}{n(1+\ln(n))^{3/2}} .$$

Compute

$$\int^{\infty} \frac{1}{n(1+\ln(n))^{3/2}} \mathrm{d}n .$$

Start by making the substition $x = \ln(n)$ where $\mathrm{d}x = (1/n)\mathrm{d}n$ such that

$$\int^{\infty} \frac{1}{(1+x)^{3/2}} \mathrm{d}x .$$

We can further simplify the integrand with the substitution $y = x + 1$ where $\mathrm{d}y = \mathrm{d}x$, to find

$$\int^{\infty} \frac{1}{y^{3/2}} \mathrm{d}y = -\frac{y^{-1/2}}{1/2}\bigg|^{\infty} = 0.$$

$\boxed{\text{Since the integral converges, the sum also converges.}}$

9. MB 13.14 Use the integral test to determine if

$$\sum_{1}^{\infty} \frac{1}{(n^2+3^2)}$$

converges or diverges. Compute

$$\int^{\infty} \frac{1}{(n^2+3^2)} \mathrm{d}n \ .$$

Let $n = 3\sinh(x)$ where $\mathrm{d}n = 3\cosh(x)\mathrm{d}x$ and we find

$$\int^{\infty} \mathrm{d}x = x\Big|^{\infty} = \sinh^{-1}(n/3)\Big|^{\infty} = \infty \ .$$

$$\boxed{\text{Since the integral diverges, so does the sum.}}$$

10. MB 15.21 Test the convergence of the sum

$$\sum_{n=0}^{\infty} \frac{5^n (n!)^2}{(2n)!} \ ,$$

using the ratio test,

$$\begin{aligned}
\rho = \lim_{n\to\infty} & \left| \frac{5^{n+1}((n+1)!)^2}{2n+1} \frac{(2n)!}{5^n (n!)^2)} \right| \\
&= \lim_{n\to\infty} \left| \frac{5(n+1)^2}{(2n+2)(2n+1)} \right| \\
&= \lim_{n\to\infty} \left| \frac{5(1+1/n)^2}{(2+2/n)(2+1/n)} \right| \\
&= \frac{5}{4} \ ,
\end{aligned}$$

and $\boxed{\text{because } p > 1\text{, the sum diverges.}}$

11. MB 15.28 Test the convergence of the sum

$$\sum_{n=0}^{\infty} \frac{n!(2n)!}{(3n)!} \ ,$$

using the ratio test,

$$\begin{aligned}
\rho = \lim_{n\to\infty} & \left| \frac{(n+1)!(2n+2)!}{(3n+3)!} \frac{(3n)!}{n!(2n)!} \right| \\
&= \lim_{n\to\infty} \left| \frac{(n+1)(2n+2)(2n+1)}{(3n+3)(3n+2)(3n+1)} \right| \\
&= \lim_{n\to\infty} \left| \frac{(1+1/n)(2+2/n)(2+1/n)}{(3+3/n)(3+2/n)(3+1/n)} \right| \\
&= \frac{4}{27} \ ,
\end{aligned}$$

and $\boxed{\text{since } \rho < 1\text{, the sum converges.}}$

12. MB 16.33 Use the special comparison test to find if the sum converges or diverges:

$$\sum_{n=5}^{\infty} a_n = \sum_{n=5}^{\infty} \frac{1}{2^n + n^2} \ .$$

Notice that series is similarly to $\sum_{n=5}^{\infty} 1/2^n$ which is a convergent geometric series. If we let $b_n = 1/2^n$ and do the special comparison test, we find

$$\begin{aligned} \lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{1}{2^n - n^2} \div \frac{1}{2^n} \\ &= \lim_{n\to\infty} \frac{1}{1 - n^2/2^n} = 1 \ . \end{aligned}$$

$\boxed{\text{Since the limit is finite, the sum converges.}}$

13. MB 16.34 Use the special comparison test to find the convergence or divergence of the sum

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{n^2 + 3n + 4}{n^4 + 7n + 6n - 3} \ ,$$

by comparing it with the series $b_n = 1/n^2$ which we know converges from Pr. 7 (MB 11.3):

$$\begin{aligned} \lim_{a\to\infty} \frac{a_n}{b_n} &= \lim_{a\to\infty} \frac{n^2 + 3n + 4}{n^4 + 7n + 6n - 3} \div \frac{1}{n^2} \\ &= \lim_{a\to\infty} \frac{1 + 3/n + 4/n^2}{n^2 + 7n + 6/n - 3/n^2} \div \frac{1}{n^2} \\ &= \lim_{a\to\infty} \frac{1 + 3/n + 4/n^2}{1 + 7/n + 6/n^3 - 3/n^4} = 1 \ . \end{aligned}$$

$\boxed{\text{Since the limit is finite, the sum converges.}}$

14. MB 17.5 The alternating series

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$$

converges if $|a_{n+1}| \leq |a_n|$ for $n > N$ where N is a finite number and the $\lim_{n\to\infty} a_n = 0$. This series satisfies both the conditions,

$$\left| \frac{1}{\ln(n+1)} \right| \leq \left| \frac{1}{\ln(n)} \right| \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{\ln(n+1)} = 0 \ ,$$

$\boxed{\text{so the series converges.}}$

15. MB 17.8 Use the alternating series test on

$$\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{10n}}{n+2} .$$

Since

$$\left| \frac{\sqrt{10}}{\sqrt{n+1} + 2/\sqrt{n+1}} \right| \leq \left| \frac{\sqrt{10}}{\sqrt{n} + 2/\sqrt{n}} \right| \quad \text{for} \quad n > 2 \quad \text{and} \quad \lim_{n\to\infty} \frac{\sqrt{10}}{\ln(\sqrt{n} + 2/\sqrt{n})} = 0 ,$$

the series converges.

16. MB 19.5 Test the convergence or divergence of the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n}{n^3 - 4} ,$$

by any means. First do the Preliminary Test

$$\lim_{n\to\infty} \frac{n}{n^3 - 4} = \lim_{n\to\infty} \frac{1/n^2}{1 - 4/n^3} = 0 ,$$

to check for divergence. Since the series passes this test, we need to test further. Let's try the root test:

$$\begin{aligned} \rho &= \lim_{n\to\infty} \left| \frac{n+1}{(n+1)^3 - 4} \div \frac{n}{n^3 - 4} \right| \\ &= \lim_{n\to\infty} \left| \frac{1 + 1/n}{(1 + 1/n)^3 - 4/n^3} \frac{1 - 4/n^3}{1} \right| = 1 . \end{aligned}$$

This test is inconclusive because $\rho = 1$, so we need to try a different test. How about we do the special comparison test with the series $b_n = 1/n^2$ which converges as shown in Pr. 7 (MB 11.3):

$$\begin{aligned} \lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \left| \frac{n}{n^3 - 4} \div \frac{1}{n^2} \right| \\ &= \lim_{n\to\infty} \left| \frac{1}{1 - 4/n^3} \right| = 1 . \end{aligned}$$

Since the limit is finite, the series converges.

17. MB 19.15 Test the convergence or divergence of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n n!}{10^n} ,$$

by any means. Use the alternating series test:

$$\left|\frac{(n+1)!}{10^{n+1}}\right| \not\leq \left|\frac{n!}{10^n}\right| \quad \text{for} \quad n \gg 1 \quad \text{and} \quad \lim_{n\to\infty} (n+1)!/10^{n+1} \to \infty \ .$$

$\boxed{\text{The alternating series fails both conditions, so the series diverges.}}$

18. MB 19.18 Test the convergence or divergence of the series

$$\sum_{n=1}^{\infty} (-1)^n / 2^{\ln(n)} \ ,$$

by any means. Use the alternating series test:

$$\left|\frac{1}{2^{\ln(n+1)}}\right| \leq \left|\frac{1}{2^{\ln(n)}}\right| \quad \text{for} \quad n \geq 1 \quad \text{and} \quad \lim_{n\to\infty} (-1)^n / 2^{\ln(n)} \to 0 \ .$$

$\boxed{\text{Both conditions are passed, so the series converges.}}$

19. MB 20.19 Test the convergence of the following series:

$$\frac{1}{2^n} - \frac{1}{3^2} + \frac{1}{2^3} - \frac{1}{3^3} + \cdots = \sum_{n=2}^{\infty} \left(\frac{1}{2^n} - \frac{1}{3^n}\right) \ .$$

Warning: Although the series can be rearranged to look like the difference between two series or two series substracted term by term, we are not allowed to rearrange the series unless it is absolutely convergent.

First we must determine if the series is absolutely convergent:

$$\sum_{n=2}^{\infty} \left(\frac{1}{2^n} + \frac{1}{3^n}\right) = \sum_{n=2}^{\infty} \frac{1}{2^n} + \sum_{n=2}^{\infty} \frac{1}{3^n} \ .$$

For a postive definite series, we are allowed to rearrange terms. In this case the series is just the sum of two convergent geometric series with the first two terms dropped. Since the addition of any two convergent series is also convergent, this means the series is absolutely convergent. And thus the alternating series also converges.

20. MB 20.21 Test the convergence of $\sum_{n=1}^{\infty} a_n$ with the recursion relation $a_{n+1} = (n/(2n+3))a_n$. Use the ratio test:

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \\
&= \lim_{n\to\infty} \left|\frac{n}{2n+3}\right| \\
&= \lim_{n\to\infty} \left|\frac{1}{2+3/n}\right| = \frac{1}{2} \ .
\end{aligned}$$

$\boxed{\text{Since } \rho < 1, \text{ the series converges.}}$

§**Recommended Supplementary problems: No scores**

S(1) MB 17.7 Find the convergence of the

$$\sum_{n=0}^{\infty} \frac{(-1)^n n}{1+n^2} \ .$$

Use the alternating series test:

$$\left|\frac{n+1}{1+(n+1)^2}\right| \le \left|\frac{n}{1+n^2}\right| \quad \text{for} \quad n \ge 1 \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n+1/n} = 0 \ ,$$

Because both conditions are satisfied, the series converges.

S(2) MB 17.10 Show that $a_n \to 0$. Explain why the following alternating series diverge using alternating series test:
a) If we group terms as follows, it appears we have the difference of two divergent series:

$$\begin{aligned} & 2 - \frac{1}{2} + \frac{2}{3} - \frac{1}{4} + \frac{2}{5} - \frac{1}{6} + \frac{2}{7} - \frac{1}{8} + \cdots \\ & = \sum_{n=1}^{\infty} \left(\frac{2}{2n-1} - \frac{1}{2n} \right) . \end{aligned}$$

We are not allowed to take the difference of two divergent series. First we test if the series is absolutely convergent. For a positive denfinte series we are allowed to rearrange terms and the sum of two divergent series is also divergent. Hence, the series does not absolutely converge. Now we must check for conditional convergence using alternating series test:

$$L = \lim_{a_n\to\infty} \left(\frac{2}{2n-1} - \frac{1}{2n} \right) \to 0;$$

and

$$\begin{aligned} & \left|\frac{1}{2n}\right| \le \left|\frac{2}{2n-1}\right| \quad \text{for odd terms} \\ & \left|\frac{2}{2n+1}\right| \not\le \left|\frac{1}{2n}\right| \quad \text{for even terms .} \end{aligned}$$

Since the series fails the inequality condition for even terms, the series diverges.

b) Similary, if we group terms as follows, it appears we have the difference of two divergent series taken term by term:

$$\begin{aligned} & \frac{1}{\sqrt{2}} - \frac{1}{2} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \frac{1}{4} - \frac{1}{\sqrt{5}} + \frac{1}{5} + \cdots \\ & = \sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{n} \right) . \end{aligned}$$

We are not allowed to take the difference of two divergent series unless the series produces a conditionally convergence. As stated in part a) the sum of two postive definite divergent series also diverges. Hence, the series does not absolutely converge. So now we must check for conditional convergence using alternating series test:

$$L = \lim_{a_n \to \infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{n} \right) \to 0$$

$$\text{and}$$

$$\left| \frac{1}{n} \right| \leq \left| \frac{1}{\sqrt{n}} \right| \quad \text{for odd terms}$$

$$\left| \frac{1}{\sqrt{n+1}} \right| \nleq \left| \frac{1}{n} \right| \quad \text{for even terms .}$$

Since the series fails the inequality condition for even terms, the series diverges.

S(3) MB 16.31 Use the special comparison test to determine if the series converges or diverges.

$$\sum_{n=9}^{\infty} \frac{(2n+1)(3n-5)}{\sqrt{n^2-73}} = \frac{6n-7-5/n}{\sqrt{1-73/n^2}}$$

Now we compare this series with the series of $b_n = 1$ which clearly must diverge:

$$L = \lim_{n \to \infty} \frac{a_n}{b_n}$$
$$= \frac{6n-7-5/n}{\sqrt{1-73/n^2}} \to \infty \; .$$

(This is actually identical to the preliminary test.)

Since the limit is infinte, the series diverges.

S(4) MB 16.37 If $L = \lim_{n \to \infty} a_n/b_n$ lies on the interval $0 < L < \infty$, we can find two positive numbers, m and M, such that $m < L < M$. We know that there exists an positive integer N such that for all $n > N$ the ratio a_n/b_n lies arbitrarily close to L so that

$$m < \frac{a_n}{b_n} < M \; .$$

Now, if we multiply through by b_n we have

$$mb_n < a_n < Mb_n \quad \text{for} \quad n > N.$$

So, if $\sum b_n$ diverges then so does $\sum mb_n$ and since $mb_n < a_n$ for all $n > N$ then the $\sum a_n$ also diverges. Similarly, if $\sum b_n$ converges then so does $\sum Mb_n$ and since $a_n < Mb_n$ for all $n > N$ then the $\sum a_n$ also converges.

S(5) MB 22.10 Find the convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x)^{2n}}{(2n)^{3/2}} ,$$

and check the endpoints for convergence or divergence. Use the ratio test:

$$\rho = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}(x)^{2n}}{(2n)^{3/2}} \div \frac{(-1)^{n+1}(x)^{2n}}{(2n)^{3/2}} \right|$$
$$= \lim_{n\to\infty} \left| \frac{(-1)x^n}{(1+1/(2n))^{3/2}} \right| = |x^2|$$

The series converges for $|x^2| < 1$. Now we check the endpoints at $x = \pm 1$ where the terms of the series are $a_n = (-1)^n/(2n)^{3/2}$. Using the alternating series test, we compute

$$\left| \frac{1}{(2n+2)^{3/2}} \right| \leq \left| \frac{1}{(2n)^{3/2}} \right|$$

and

$$\lim_{n\to\infty} \frac{1}{(2n)^{3/2}} \to 0 ,$$

which satisfies both conditions, so the seris is convergent at $x = \pm 1$.

Thus the power series is convergent for $|x| \leq 1$.

S(6) MB 22.13 Find the convergence of the power series

$$\sum_{n=1}^{\infty} \frac{n(-x)^n}{n^2+1} ,$$

and check the endpoints for convergence or divergence. Use the ratio test:

$$\rho = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}(n+1)(x)^{n+1}}{(n+1)^2+1} \div \frac{n(-1)^n x^n}{n^2+1} \right|$$
$$= \left| \frac{(-1)x(1+1/n)1+1/n^2)}{(1+1/n)^2+1/n^2} \right| = |x|$$

The series converges for $|x| < 1$. Now we check the endpoints, starting with $x = 1$ where the terms of the series are $a_n = (-1)^n n/(n^2+1)$. Using the alternating series test, we find

$$\left| \frac{n+1}{(n+1)^2+1} \right| \leq \left| \frac{n}{n^2+1} \right|$$

and

$$\lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{1}{n+1/n} = 0 .$$

and thus the series converges for $x = 1$. For $x = -1$ the terms of the series are $a_n = n/(n^2 + 1)$. Using the special comparison test for $b_n = 1/n$, we find

$$\begin{aligned} \lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{n}{n^2+1} \div \frac{1}{n} \\ &= \lim_{n\to\infty} \frac{1}{1+1/n^2} = 1 \, . \end{aligned}$$

Since the limit exists and the $\sum b_n$ diverges, the $\sum a_n$ also diverges.

Hence, the power series is convergent for $-1 < x \leq 1$.