Mathematical Methods of Physics 116A- Winter 2018

Physics 116A

Home Work # 2 Solutions
Posted on Jan 18, 2018
Due in Class Jan 25, 2018

§**Required Problems: Each problem has 5 points**

E.g. MB 19.16 means problem #16 on page 19 in the book by M. Boas, 3rd Edition.

1. Let's make these oranges disguishable by numbering them 1 through N and placing them in a row, i.e. $(1, 2, \cdots, N)$. There are $N!$ ways to permutate these oranges. For example we can represent the number of permutations for $N = 3$ as

$$3! = \{(1,2,3),(2,3,1),(3,1,2),(2,1,3),(1,3,2),(3,2,1)\} = 6 ,$$

where we can see there are 6 ways of ordering the oranges in a row by counting the number of elements in the curly brackets. Using this we can show there are $N!/(N-M)!$ ways of choosing the first $M \leq N$ oranges to share with friends. Notice that the factorial $(N-M)!$ reduces all permutations where the first M elements are in identical order, but rest are in a different order, down to a single choice. In the case $N = 4$ and $M = 2$, we say that $(1,2,3,4) \equiv (1,2,4,3)$, since both give the same choice, $(1,2)$. Now we can represent the number of possible choices as $N!/(N-M)!$ which at $N = 3$ for $M = 2$ is

$$\frac{3!}{(3-2)!} = \{(1,2),(2,3),(3,1),(2,1),(1,3),(3,2)\} = 6 .$$

However, the order in which we remove the oranges from the crate to share does not matter, i.e. $(1,2) \equiv (2,1)$. So we can further reduce the number ways of choosing the first $M \leq N$ oranges by $M!$, where $M!$ accounts for the number of different permutations we can remove the same set of M distugishable oranges from the crate. Finally, we see that the number of possible choices, $\boxed{R, \text{ can be represented as } N!/(M!(N-M)!)}$.

For example, in the case $N = 3$ for all possible M we have

$$\binom{3}{0} = \frac{3!}{0!(3-0)!} = \{\varnothing\} = 1 \tag{1}$$

$$\binom{3}{1} = \frac{3!}{1!(3-1)!} = \{(1),(2),(3)\}\} = 3 \tag{2}$$

$$\binom{3}{2} = \frac{3!}{2!(3-2)!} = \{(1,2),(2,3),(3,1)\} = 3 \tag{3}$$

$$\binom{3}{3} = \frac{3!}{3!(3-3)!} = \{(1,2,3)\} = 1 . \tag{4}$$

We see that there is only one way to share zero oranges. There are three ways of sharing one orange. There are three ways of sharing two oranges, and there is only one way to share all the oranges.

2. Using the general Maclaurin series formula, expand out

$$(1+x)^\alpha = \sum_{j=0}^{\infty} A_j x^j \ .$$

For a Maclaurin series the coefficent are defined as $A_j = f^{(j)}(0)/j!$. Next we compute $f^{(j)}(0)$:

$$f^{(0)}(0) = (1+x)^\alpha \Big|_{x=0} = 1$$

$$f^{(1)}(0) = \alpha(1+x)^{\alpha-1} \Big|_{x=0} = \alpha$$

$$f^{(2)}(0) = \alpha(\alpha-1)(1+x)^{\alpha-2} \Big|_{x=0} = \alpha(\alpha-1)$$

$$f^{(3)}(0) = \alpha(\alpha-1)(\alpha-2)(1+x)^{\alpha-3} \Big|_{x=0} = \alpha(\alpha-1)(\alpha-2)$$

$$f^{(4)}(0) = \alpha(\alpha-1)(\alpha-2)(\alpha-3)((1+x)^{\alpha-4} \Big|_{x=0} = \alpha(\alpha-1)(\alpha-2)(\alpha-2)(\alpha-3)$$

$$f^{(5)}(0) = \alpha(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)(1+x)^{\alpha-5} \Big|_{x=0} = \alpha(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)$$

$$\vdots$$

$$f^{(n)}(0) = \alpha(\alpha-1)\cdots(\alpha-n+1)(1+x)^{\alpha-n} \Big|_{x=0} = \alpha(\alpha-1)\cdots(\alpha-n+1) \ .$$

In general $\alpha(\alpha-1)\cdots(\alpha-n+1) = \Gamma(\alpha+1)/\Gamma(\alpha+1-n)$ for all $\alpha \in \mathbb{R}$ and $\Gamma(j+1) = j!$ for all integers $j \geq 0$. Hence,

$$\boxed{(1+x)^\alpha = \sum_{j=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(j+1)\Gamma(\alpha+1-j)} x^j \ .}$$

For $\alpha = 4$ the coefficent A_j is a constant, $f^{(4)}(x) = \alpha(\alpha-1)(\alpha-2)(\alpha-3)/3!$, and thus all higher order derivatives are zero. Hence, the series truncates at order x^4.

3. MB 40.6 Use Maclaurin series to evalute

$$\frac{d^3}{dx^3}\left(\frac{x^2 e^x}{1-x}\right)\Big|_{x=0} \ .$$

The first step is to recognize that $1/(1-x) = \sum_{n=0}^{\infty} x^n$ is a geometric series with a radius of convergence of $|x| < 1$ and $e^x = \sum_{n=0}^{\infty} x^n/n!$ which is convergent for all $x \in \mathbb{R}$. We expand both series only to first order (because all higher order terms drop out when we set $x = 0$) and differentiate

$$x^2(1 + x/1!)(1 + x) = x^2 + 2x^3 + x^4$$

$$\frac{d^3}{dx^3}(x^2 + 2x^3 + x^4)\bigg|_{x=0} = \boxed{12}\,.$$

4. MB 41.12 Use Maclaurin series to evalute

$$\lim_{x\to 0} \frac{\tan x - x}{x^3}\,.$$

The series for $\tan x = x + x^3/3 + 2x^5/15 + \cdots$ is a little more complicated than other trig functions, fortuantely we only need the first few terms since all higher order terms drop out as $x \to 0$.

$$\lim_{x\to 0} \frac{(x + x^3 2 + x^5/15) - x}{x^3} = \lim_{x\to 0}\left(\frac{1}{3} + \frac{2x^2}{15}\right) = \boxed{\frac{1}{3}}\,.$$

5. MB 41.14 Find a two term approximation for following integral and an error bound for the given t interval:

$$\int_0^t e^{-x^2}\mathrm{d}x\,, \quad 0 < t < 0.1\,.$$

The Maclaurin series expansion of $e^y = 1/0! + y/1! + y^2/2! + \cdots$. We can compose any series with an polynomial or another another series where their radius of convergence overlaps such that $e^{-x^2} = 1 - x^2/1! + x^4/2! + \cdots$. Thus the two term approximation of the integral with the error in ellipses is given by

$$\int_0^t e^{-x^2}\mathrm{d}x \approx \int_0^t \mathrm{d}x - \int_0^t x^2\mathrm{d}x + \left(\int_0^t x^4\mathrm{d}x\right) = \boxed{t - t^3/3 + (t^5/10)}\,.$$

Since this is an alternating series, the error bound is estimated by the next higher order term, $t^5/5$, which has a maximum value of $\boxed{10^{-6}}$ on the t interval.

6. MB 41.18 Find the sum of the following series by recognizing it as a Maclaurin series:

$$\sum_{n=1}^{\infty} \frac{1}{n2^n} \, .$$

Note that $\ln(1-x) = \sum_{n=1}^{\infty} -x^n/n$, so if we let $x = 1/2$, we find $-\sum_{n=1}^{\infty} 1/2^n/n$. $\boxed{\text{Hence, } \sum_{n=1}^{\infty} 1/(n2^n) = -\ln(1-1/2) = \ln(2).}$

7. MB 42.29

a) Find F/W as a series of power of θ

$$\begin{aligned} \frac{F}{W} &= \frac{T\sin\theta}{T\cos\theta} \\ &= \tan\theta \\ &= \boxed{\theta - \frac{\theta^3}{3} + \frac{2\theta^5}{15} + \cdots} \end{aligned}$$

b) Find F/W as a series in powers of x/l.

$$\begin{aligned} \frac{F}{W} &= \frac{\sin\theta}{\cos\theta} \\ &= \frac{\sin\theta}{\sqrt{1-\sin^2\theta}} \, , \quad \text{where } \sin\theta = x/\ell \\ &= \frac{x^2/\ell^2}{\sqrt{1-x^2/\ell^2}} \\ &= y\sum_{n=0}^{\infty} \binom{-1/2}{n} y^n \, , \quad \text{where } y = x^2/\ell^2 \\ &= \boxed{x/\ell + \frac{x^3/\ell^3}{2} + \frac{3x^5/\ell^5}{8} + \frac{5x^7/\ell^7}{16} + \frac{35x^9/\ell^9}{128} + \cdots} \end{aligned}$$

We expand one over the denomiator as a binomial series composed of the polynomial x^2/ℓ^2 with $\alpha = -1/2$ and the resulting series has an interval of convergence $|x| < 1$.

8. MB 45.12 Find the interval of convergence, including the endpoints:

$$\sum_{n=1}^{\infty} \frac{x^n n^2}{5^n(n^2+1)} \, .$$

Use the ratio test:

$$\begin{aligned}\rho &= \lim_{n\to\infty}\left|\frac{x^{n+1}(n+1)^2}{5^{n+1}((n+1)^2+1)} \div \frac{x^n n^2}{5^n(n^2+1)}\right| \\ &= \lim_{n\to\infty}\left|\frac{x(1+1/n)(1+1/n^2)}{5((1+1/n)^2+1)}\right| \\ &= \left|\frac{x}{5}\right| .\end{aligned}$$

Thus series is convergent on the interval $|x| < 5$. Next we need to check the endpoints. At $x = 5$ the series is $a_n = n^2/(n^2+1)$ and if use the prelimary test,s we find that $\lim_{n\to\infty} n^2/(n^2+1) = 1$, so the series diverges. At $x = -5$ we have an alternating series, $a_n = (-1)^n n^2/(n^2+1)$. As show above the $\lim_{n\to\infty} a_n \neq 0$, so the series diverges by the alternating series test. $\boxed{\text{Hence, } |x| < 5.}$

9. MB 45.23 Use series you know to show that:

$$\frac{\pi^2}{3!} - \frac{\pi^4}{5!} + \frac{\pi^6}{7!} - \cdots = 1 .$$

This series looks very similar to $\sin x = \sum_{n=0}^{\infty}(-1)^{n+1}x^{2n+1}/(2n+1)!$ which at $x = \pi$:

$$\begin{aligned}\sin\pi &= \frac{\pi}{1!} - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \cdots = 0 , \\ \frac{\sin\pi}{\pi} &= 1 - \frac{\pi^2}{3!} + \frac{\pi^4}{5!} - \frac{\pi^6}{7!} + \cdots = 0 ,\end{aligned}$$

$$\boxed{\frac{\pi - \sin\pi}{\pi} = \frac{\pi^2}{3!} - \frac{\pi^4}{5!} + \frac{\pi^6}{7!} - \cdots = 1 .}$$

10. Simplify the following into the form $re^{i\theta}$ or $x + iy$ and plot:

a) MB 52.11 Given $z = 17 - i12$ compute $r = \sqrt{x^2+y^2} = \sqrt{433}$ and $\theta = \arctan y/x \approx -0.614$ and write $\boxed{z \approx \sqrt{433}e^{-i0.614}}$ and plot Fig. 1.

b) MB 52.11 Given $3(\cos 28^\circ + i\sin 28^\circ)$ note that $r = 3$ and compute $\theta = 28^\circ \times (2\pi/360^\circ) = 7\pi/45$ or compute $x = 3\cos(7\pi/45) \approx 2.648$ and $y = 3\sin(7\pi/45) \approx 1.408$ and write $\boxed{z = 3e^{i7\pi/45} \text{or } z \approx 2.648 + i1.408}$ plot Fig. 2.

§**Recommended Supplementary problems: No scores**

S(1) MB 51. 1-9

S(2) MB 52. 1-8

For solutions to the supplmentary problems, come to the scheduled TA office hours or make an appointment with the TA via email.

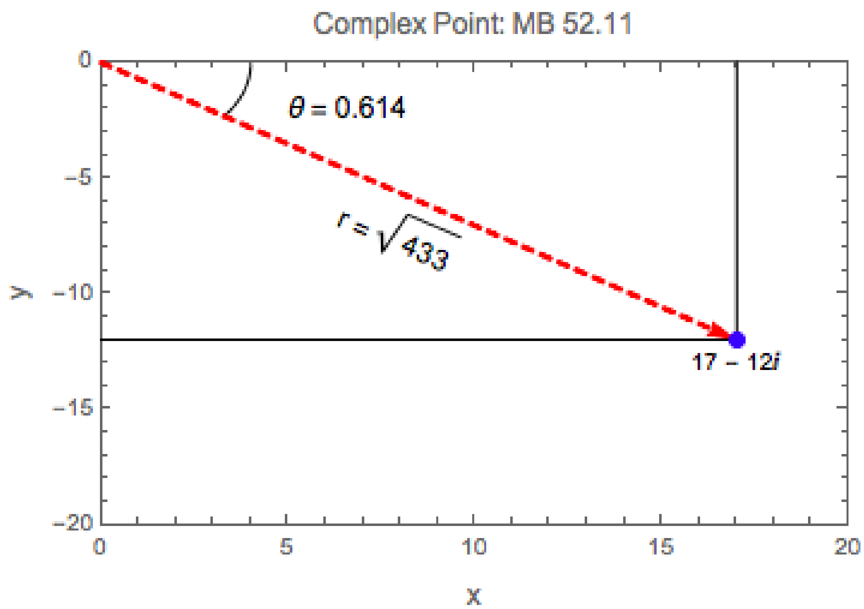

Figure 1:

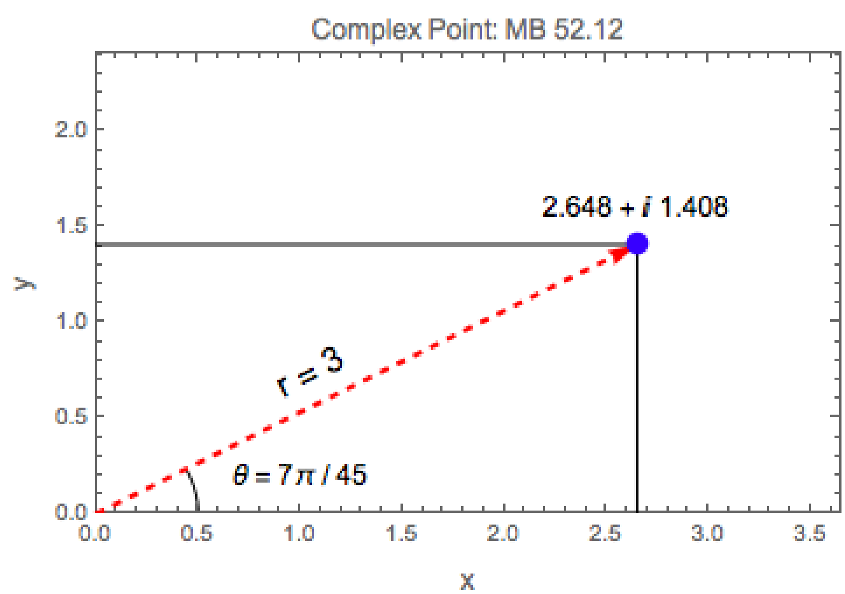

Figure 2: