Mathematical Methods of Physics 116A- Winter 2018

Physics 116A

Home Work # 3 Solutions
Posted on Jan 25, 2018
Due in Class Feb 1, 2018

§**Required Problems: Each problem has 5 points**

E.g. MB 19.16 means problem #16 on page 19 in the book by M. Boas, 3rd Edition.

1. MB 53.23 Find

$$f(z) = \frac{1+z}{1-z}$$

in rectangular $(x+iy)$ form if case a) $z = 2-3i$ and if case b) $z = x+iy$:

a)

$$\begin{aligned} f(2-3i) &= \frac{3-3i}{-1+3i} \\ &= \frac{(3+3i)}{(-1+3i)}\frac{(-1-3i)}{(-1-3i)} \\ &= \boxed{-\frac{6+3i}{5}} \end{aligned}$$

b)

$$\begin{aligned} f(x+iy) &= \frac{1-x+iy}{1-x-iy} \\ &= \frac{(1+x+iy)}{(1-x+iy)}\frac{(1-x+iy)}{(1-x+iy)} \\ &= \boxed{\frac{1-x^2-y^2}{(1-x)^2+y^2} + i\frac{2y}{(1-x)^2+y^2}} \end{aligned} \quad (1)$$

2. Find the absolute value of each of the following:

a) MB 53.30 Given

$$z = \frac{3i}{i-\sqrt{3}} \ ,$$

we find absolute value by taking $\sqrt{z\bar{z}}$, where $\bar{z}$ is the complex conjugate, i.e. setting $i \to -i$. Thus $\bar{z} = 3i/(\sqrt{3}+i)$, so

$$\sqrt{z\bar{z}} = \sqrt{\left(\frac{3i}{i-\sqrt{3}}\right)\left(\frac{3i}{i+\sqrt{3}}\right)} = \boxed{\frac{3}{2}}$$

b) MB 53.32

Similarly, given

$$z = (2 + 3i)^4 \ ,$$

the absolute value is

$$\sqrt{z\bar{z}} = \sqrt{(2+3i)^4(2-3i)^4} = [(2+3i)(2-3i)]^2 = [13]^2 = \boxed{169.}$$

3. MB 55.62 Describe geometrically the set of points in the complex plane satisfying the equation

$$|z + 1| + |z - 1| = 8.$$

In order to see the geometry of this expression, it is helpful to put it in the more familar rectangular (x + iy) form:

$$|(x + 1) + iy| + |(x - 1) + iy| = 8 \ .$$

Now, if we compute the absolute value, we find

$$\sqrt{(x + 1)^2 + y^2} + \sqrt{(x - 1) + y^2} = 8 \ .$$

This is the equation of ellipse with foci at $c = (-1, 0)$ and $(1, 0)$, and a semi-major axis of $a = 4$. The expression states that an arbitary point (x, y) on the ellipse must satify the condition that the sum of the distances from each foci to the arbitary point be equal to twice the length of semi-major axis.

4. MB 56.68

Find the speed (v) and magnitude of acceleration (a) if $z = \cos(2t) + i\sin(2t)$ and describe the motion. Let's start by taking the first and second derivatives of z:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = -2\sin^2(2t) + 2i\cos(2t) = -2iz$$

and

$$\frac{\mathrm{d}^2z}{\mathrm{d}t^2} = -4\cos(2t) - 4i\sin(2t) = -4z \ .$$

The speed is

$$v = \left|\frac{\mathrm{d}z}{\mathrm{d}t}\right| = \sqrt{4\sin^2(2t) + 4\cos^2(2t)} = \boxed{2}$$

and the magnitude of the acceleration is

$$a = \left|\frac{\mathrm{d}^2z}{\mathrm{d}t^2}\right| = \sqrt{16\cos^2(2t) + 16\sin^2(2t)} = \boxed{4} \ .$$

The particle moves in a uniform circle of radius 1 centered about the origin, the tangential velocity of the particle is 2 and centripetal acceleration is 4.

5. MB 59.7 Find the convergence of the complex power series

$$\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} \, .$$

Use the ratio test:

$$\begin{aligned}
\rho = & \lim_{n\to\infty} \left| \frac{z^{2n+2}}{(2n+2)!} \div \frac{z^{2n}}{(2n)!} \right| \\
= & \lim_{n\to\infty} \left| \frac{z^2}{(2n+2)(2n+1)} \right| \\
= & \lim_{n\to\infty} \left| \frac{(z/n)^2}{(2+2/n)(2+1/n)} \right| = 0 \, .
\end{aligned}$$

$\boxed{\text{Since } \rho = 0 < 1, \text{ the series is convergent for all } |z| < \infty.}$

6. Express the following complex numbers in the form $x + iy$.

a) MB 63.15 Given that $z = (1+i)^2 + (1+i)^4$, we can simpify this expression by putting $1+i$ into polar form. This can be done inour head if we recognize that $1+i$ makes $45-45-90$ triangle with hypotenuse of 2, i.e. $r = 2$ and $\theta = \pi/4$:

$$\begin{aligned}
z &= \left(\sqrt{2}e^{i\pi/4}\right)^2 + \left(\sqrt{2}e^{i\pi/4}\right)^4 \\
&= 2e^{i\pi/2} + 4e^{i\pi} \\
&= 2i - 4 = \boxed{-4+2i} \, .
\end{aligned}$$

b) MB 63.18 Given that $z(1+i)/(1-i)$, similarly to the previous problem we can simplify this expression by putting $1+i$ and $1-i$ into polar form. This can be done quickly by recognizing that $1-i$ is also a $45-45-90$ triangle with hypotenuse of 2; however, this time $r = 2$ and $\theta = -\pi/4$:

$$\begin{aligned}
z &= \left(\frac{2e^{i\pi/4}}{2e^{-i\pi/4}}\right)^4 \\
&= \left(e^{i\pi/2}\right)^4 \\
&= e^{2\pi} = \boxed{1} \, .
\end{aligned}$$

7. MB 67.31 Show that the sum of the N roots of the Nth root of any complex number z is zero. The N roots of z can be represented as

$$z^{1/N} = r^{1/N} e^{i\theta/N} e^{i2\pi m/N} \quad \text{for all integers } m \in [0, N-1].$$

The sum of the N roots is expressed as

$$S_N \equiv r^{1/N} e^{i\theta/N} \sum_{m=0}^{N-1} e^{i2\pi m/N} \ .$$

The sum is a partial geometric series:

$$r^{1/N} e^{i\theta/N} \sum_{m=0}^{N-1} w^m = a\frac{1-w^N}{1-w} \ ,$$

where $w = e^{i2\pi/N}$ and $a = r^{1/N}e^{i\theta/N}$. We note that $w^N = e^{i2\pi} = 1$, hence

$$\boxed{S_N = a\frac{1-w^N}{1-w} = 0 \ ,}$$

unless $N = 1$ in which case $S_1 = a$.

8. MB 69.18 Evaluate $\int e^{(a+ib)x} \mathrm{d}x$ and take the imaginary part to show that

$$\int e^{ax} \sin bx dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2} \ .$$

Using Euler's formula ($e^{i\theta} = \cos\theta + i\sin\theta$) we can show that

$$\mathrm{Im}\left\{ \int e^{(a+ib)x} \right\} = \int e^{ax} \sin bx dx \ .$$

Integrate

$$\begin{aligned}
\int e^{(a+ib)x} \mathrm{d}x &= \frac{e^{(a+ib)x}}{a+ib} \\
&= \frac{e^{(a+ib)x}}{a+ib}\frac{a-ib}{a-ib} \\
&= \frac{e^{ax}(\cos bx + i\sin bx)(a-ib)}{a^2+b^2} \\
&= e^{ax}[(a\cos bx + b\sin bx) + i(a\sin bx - b\cos bx)] \ .
\end{aligned} \tag{2}$$

Take the imaginary part of Eq. (2):

$$\boxed{\mathrm{Im}\left\{ \int e^{a+ib} \mathrm{d}x \right\} = e^{ax}(a\sin bx - b\cos bx)} \ .$$

9. MB 77.4 In this problem, $z(t)$ represents the path of particle on the (x, y) plane as a function of time. Find the speed, the magnitude of the acceleration and describe the motion of the function given by

$$z(t) = (1+i)t - (2+i)(1-t) \ .$$

We can show that the path is linear by expressing the function in parameteric form

$$z(t) = (3t - 2, 2t - 1) \ ,$$

where the slope $m = 2/3$ and the initial position of the particle is $(-2, -1)$. The speed is

$$\left| \frac{\mathrm{d}z}{\mathrm{d}t} \right| = |(1 + i) - (2 + i)| = \sqrt{13} \ ,$$

and the magnitude of the acceleration is

$$\left| \frac{\mathrm{d}^2 z}{\mathrm{d}t^2} \right| = 0 \ .$$

10. MB 79.11

Prove that

$$\cos\theta + \cos 3\theta + \cos 5\theta + \cdots + \cos(2n-1)\theta = \frac{\sin 2n\theta}{2\sin\theta} \ , \tag{3}$$

and

$$\sin\theta + \sin 3\theta + \sin 5\theta + \cdots + \sin(2n-1)\theta = \frac{\sin^2 2n\theta}{\sin\theta} \ . \tag{4}$$

Using Euler's formula we can compactly express the above two series as the expression

$$S \equiv e^{\theta} + e^{3\theta} + e^{5\theta} + \cdots + e^{(2n-1)\theta} \ ,$$

where the Re$\{S\}$ is equal to Eq. (3) and the Im$\{S\}$ is equal to Eq. (4). This geometric progression can be written in the form of MB Eq. 16.17, where $t = \theta$ and $\delta = 2\theta$. From MB Eq. 16.20, we have

$$S \equiv e^{i\{t+(n-1)\delta/2\}} \frac{\sin n\delta/2}{\sin \delta/2} \ . \tag{5}$$

Next, we plug $t = \theta$ and $\delta = 2\theta$ back into Eq. (5) to find

$$\begin{aligned} S &= e^{i2n\theta} \frac{\sin n\theta}{\sin\theta} \\ &= (\cos 2n\theta + i \sin 2n\theta) \frac{\sin n\theta}{\sin\theta} \ . \end{aligned}$$

Using the trigonometric identity $\sin 2\theta = 2\sin\theta\cos\theta$, we can conclude that

$$\mathrm{Re}\{S\} = \cos\theta + \cos 3\theta + \cos 5\theta + \cdots + \cos(2n-1)\theta = \boxed{\frac{\sin 2n\theta}{2\sin\theta}} \ ,$$

and

$$\mathrm{Im}\{S\} = \sin\theta + \sin 3\theta + \sin 5\theta + \cdots + \sin(2n-1)\theta = \boxed{\frac{\sin^2 n\theta}{\sin\theta}} \ .$$

§**Recommended Supplementary problems: No scores**

S(1) MB 74.22

Evalute the following in rectangular form $(x + iy)$ and compare with a computer solution:

$$w = \sin\left[i \ln\left(\frac{\sqrt{3}+i}{2}\right)\right] . \tag{6}$$

The simpliest way to evaluate Eq. (6) is to put $z = \frac{\sqrt{3}+i}{2}$ in polar form. We can do this in our head if we recognize that z forms a $30-60-90$ triangle in the complex plane, where $r = 1$ and $\theta = \pi/6 + 2\pi n$ for all $n \in \mathbb{Z}$. Next we take the natural log to get

$$\ln\left(e^{i(\pi/6+2\pi n)}\right) = i(\pi/6 + 2\pi n)$$

Notice that since z in polar form is periodic, there are an infinite number of ways of expressing the same complex-value z. In order to make the function of z one to one, we must perfrom a branch cut, where a branch cut is a restrict of the domain of z to a single period such that every complex-value in the target is mapped only once. However, we need not concern ourselves with complication since

$$\sin(-\pi/6 - 2\pi n) = \boxed{-1/2} ,$$

for all branches of $\ln(z)$. However, a computer program requires that we choose a branch cut for the natural log function. The branch cut of the natural log in Mathematica is choosen to be along the negative real axis, i.e. $\theta \in [\pi, -\pi)$.

S(2) MB 74.23 Evalute the following in rectangular form $(x + iy)$ and compare with a computer solution:

$$w = (1 - \sqrt{2i})^i .$$

The first step would be to compute the square root by putting $z = e^{i\theta+2\pi n}$ into polar from. In this case there are two unique roots of z given by $\sqrt{2}e^{i\pi/4+n\pi}$ for $n = 0, 1$. Notice that $e^{i\pi n} = (-1)^n$, and $\sqrt{2}e^{i\pi/4} = 1 + i$ so that $z = (1 + i)(-1)^n$. Using the relation

$$a^b = e^{b\ln(a)}$$

we can re-express Eq. (2) as

$$e^{i\ln(1-z)} .$$

Now we can plug in n and z for the two cases and solve to find

$$w = \begin{cases} e^{\pi/2} & \text{if } n = 0 \\ e^{i\ln(5)/2}e^{-\arctan[1/2]} & \text{if } n = 1 \end{cases} .$$

S(3) MB 77.5 In this problem, $z(t)$ represents the path of particle on the (x, y) plane as a function of time. Find the speed, the magnitude of the acceleration and describe the motion of the function given by

$$z(t) = z_1 t - z_2(1 - t) \ .$$

We can show that the path is linear first expressing z_1 and z_2 in rectangular form as $z_n = x_n + iy_n$ and then we can expressing the function in parameteric form

$$z(t) = (x_1 - x_2)t + x_1, (y_1 - y_2)t + y_2) \ ,$$

where the slope $m = (y_1 - y_2)t/(x_1 - x_2)t$ and the initial position of the particle is (x_2, y_2). The speed is

$$\begin{aligned} \left|\frac{\mathrm{d}z}{\mathrm{d}t}\right| &= |z_1 - z_2| \\ &= \sqrt{(z_1 - z_2)(\bar{z_1} - \bar{z_2})} \\ &= z_1\bar{z_1} - \bar{z_2}z_2 - z_1\bar{z_2} - z_2\bar{z_1} \ , \end{aligned}$$

and the magnitude of the acceleration is

$$\left|\frac{\mathrm{d}^2 z}{\mathrm{d}t^2}\right| = 0 \ .$$

S(4) MB 81.28 Evaluate the following absolute square of a complex number. Assume a and b are real. Express your answer in term of a hyperbolic function.

$$\left|\frac{(a + ib)e^b - (a - ib)^2 e^{-b}}{4abie^{-ia}}\right|^2 \tag{7}$$

The first step is to expand $(a+bi)^2$ and $(a-bi)^2$ in the numerator and then group terms using the relations $2\cosh b = e^b + e^{-b}$ and $2i\sinh b = e^b - e^{-b}$ such that Eq. (7) becomes

$$\left|\left(\cosh(b)e^{-ia} + \frac{b^2 - a^2}{2ab}\sinh(b)i\right)e^{ia}\right|^2 \ .$$

We can simplify the algebra in Eq. (4) by setting $A = \cosh b$ and $B = (b^2 - a^2)/(2ab)\sinh b$:

$$|(A + Bi)e^{ia})|^2$$

Expanding the above expression and taking the absolute value gives

$$(A\cos a - B\sin a)^2 + (B\cos a + A\sin a)^2 \ .$$

Next we can expand the expression and cancel like terms find

$$A^2 + B^2 = \cosh^2 b + \frac{(b^2 - a^2)}{2ab} \sinh^2 b \, .$$

We can put this in terms a single hyperbolic function using the hyberbolic identity $\cosh^2 b - \sinh^2 b = 1$ such that

$$1 + \left(\frac{2ab}{2ab}\right)^2 \sinh^2 b + \left(\frac{(b^2 - a^2)}{2ab}\right)^2 \sinh^2 b \, .$$

If we do a little algebra magic, we can show that $(b^2 - a^2)^2 + (2ab)^2 = (b^2 + a^2)$, so a solution simplified to a single hyperbolic term is

$$\boxed{1 + \left(\frac{b^2 + a^2}{2ab}\right) \sinh^2 b} \, .$$

S(5) MB 81.25 Prove the following:

a) Show that $\overline{\cos z} = \cos \bar{z}$.

Note that $\bar{z}$ is the complex conjugate of z and we can find the complex conjugate of a complex number by setting $i \to -i$. Let's set $w = \cos z$. Using Euler's idenity for $\cos z$ we have

$$\begin{aligned} \bar{w} &= \overline{\left(\frac{e^{iz} + e^{-iz}}{2}\right)} \\ &= \left(\frac{e^{-i\bar{z}} + e^{i\bar{z}}}{2}\right) \\ &= \boxed{\cos \bar{z}} \end{aligned}$$

b) Is $\overline{\sin z} = \sin \bar{z}$? Let $w = \sin z$ and use Euler identity for $\sin z$ to check that

$$\begin{aligned} \bar{w} &= \overline{\left(\frac{e^{iz} - e^{-iz}}{2i}\right)} \\ &= \left(\frac{e^{-i\bar{z}} - e^{i\bar{z}}}{-2i}\right) \\ &= \sin \bar{z} \end{aligned}$$

$\boxed{\text{Yes, } \overline{\sin z} = \sin \bar{z}.}$

c) If $f(z) = 1 + iz$ then is $\overline{f(z)} = f(\bar{z})$? Let's check:

$$\overline{f(z)} = \overline{1 + iz} = 1 - i\bar{z} \, .$$

$\boxed{\text{No, because } 1 - i\bar{z} \neq 1 + i\bar{z}}$

d) If $f(z)$ is expanded in a power series with real coefficients (c_n), show that $\overline{f(z)} = f(\bar{z})$.

Note that $\overline{c_n z^n} = c_n \bar{z}^n$ because the coefficients are real and complex conjugate operator is distributive, i.e. $\overline{z_1 + z_2} = \bar{z_1} + \bar{z_2}$, where z_1 and z_2 are complex numbers. Hence,

$$\boxed{\overline{f(z)} = \overline{\sum_{n=0}^{\infty} c_n z^n} = \sum_{n=0}^{\infty} c_n \bar{z}^n = f(\bar{z})\ .}$$

e) Using part (d), without computing it's value, show that $g(z) = i[\sinh(1+i) - \sinh(1-i)]$ is real.

Let $f(z) = \sinh(z)$ and $z = 1 + i$, such that $g(z) = i[f(z) - f(\bar{z})]$. Note that if $\overline{g(z)} = g(z)$, then $g(z)$ is real. Let's check:

$$\overline{g(z)} = \overline{i[f(z) - f(\bar{z})]} = -i[f(\bar{z}) - f(z)] = g(z\ .)$$

Hence, $g(z)$ is real.

S(6) MB 81. 17 to 24

Solutions to the remaining exercises are available upon request during the TA office hours or the discussion sections.