Mathematical Methods of Physics 116A- Winter 2018

Physics 116A

Home Work # 4 Solutions
Posted on Jan 1, 2018
Due in Class Feb 8, 2018

§**Required Problems: Each problem has 5 points**

E.g. MB 19.16 means problem #16 on page 19 in the book by M. Boas, 3rd Edition.

1. MB 78.7 Find the impedience for Z_1 and Z_2 in series and in parallel in the following:

a) Given $Z_1 = 1 - i$, $Z_2 = 3i$ the impedence in series is

$$Z = Z_1 + Z_2 = 1 - i + 3i = \boxed{1 + 2i}\ .$$

The impedence for Z_1 and Z_2 in parallel is

$$\begin{aligned}\frac{1}{Z} &= \frac{1}{Z_1} + \frac{1}{Z_2} \\ &= \frac{Z_1 + Z_2}{Z_1 Z_2}\ ,\end{aligned}$$

and now we take the inverse to find

$$\begin{aligned}Z = \frac{Z_1 Z_2}{Z_1 + Z_2} &= \frac{(1-i)(3i)}{1+2i} \\ &= \frac{(3i-3)}{1+2i}\frac{(1-2i)}{(1-2i)} \\ &= \boxed{\frac{9-3i}{5}}\ .\end{aligned}$$

There is also a trivial solution at $\omega = 0$ which cooresponds to a direct current.

b) Given $|Z_1| = 3.16$, $\theta_1 = 18.4°$ and $|Z_2| = 4.47$, $\theta_2 = 63.4°$ the impedence in series is

$$Z = Z_1 + Z_2 = 3.16(\cos(18.4) + i\sin(18.4)) + 4.47(\cos(63.4) + i\sin(63.4))$$

$$\begin{aligned}Z &= Z_1 + Z_2 \\ &= (3.16\cos(18.4) + 4.47\cos(63.4)) + i(3.16\sin(18.4) + 4.47\sin(63.4)) \\ &= \boxed{4.999 + i4.994}\ .\end{aligned}$$

The impedence in parallel is

$$\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2} \ .$$

The polar form we can easily take the argument into the numerator:

$$\begin{aligned}\frac{e^{-i\theta_1}}{|Z_1|} + \frac{e^{-i\theta_2}}{|Z_2|} &= \left(\frac{\cos(-18.4)}{3.16} + \frac{\cos(-63.4)}{4.47}\right) \\ &+ i\left(\frac{\sin(-18.4)}{3.16} + \frac{\sin(-63.4)}{4.47}\right) \\ &= 0.400447 - i0.299923 \ .\end{aligned}$$

The inverse is

$$\boxed{Z = 1.599 + i1.199} \ .$$

2. MB 78.8

   Find the impedence of a circuit with R and L in series and then C in parallel with them. The impedence of a resistor, inductor and capacitor are $Z_R = R$, $Z_L = i\omega L$, and $Z_C = 1/(i\omega C)$ respectively. For this circuit the total impedence is given by

   $$\frac{1}{Z} = \frac{1}{Z_R + Z_L} + \frac{1}{Z_C} \ .$$

   After doing some algebra we can re-express this as

   $$Z = \frac{(Z_R + Z_L)Z_C}{Z_R + Z_L + Z_C} \ .$$

   Next we plug in $Z_R = R$, $Z_L = i\omega L$, and $Z_C = 1/(i\omega C)$:

   $$Z = \frac{(R + i\omega L)/(i\omega C)}{R + i\omega L + 1/(i\omega C)} \ .$$

   Now we remove the complex number from the denominator

   $$Z = \frac{(-iR/(\omega C) + L/C)(R - i(\omega L - 1/(\omega C)))}{R^2 + (\omega L - 1/(\omega C))^2} \ .$$

   Next we expand the numberator and combine like terms to find

   $$Z = \frac{R/(\omega^2 C^2) + i(L/(\omega C^2) - (\omega L^2)/C - R^2/(\omega C))}{R^2 + (\omega L - 1/(\omega C))^2} \ .$$

   Finally, we can simplify the expression by multipling the numerator and bottom by $\omega^2 C^2$

   $$Z = \boxed{\frac{R + i(L\omega - R^2 C\omega - L^2 C\omega^3)}{R^2\omega^2 C^2 + (\omega^2 LC - 1)^2}} \ .$$

Find ω in terms of R, L, C at resonance. Resonance occurs when the impedence is entirely real, so when

$$L^2C\omega^3 - L\omega + R^2C\omega = 0 ,$$

we have resonance. Now we can solve for ω^2 to find

$$\boxed{\omega^2 = \frac{L - R^2}{L^2C} = \frac{1}{LC}\left(1 - \frac{R^2C}{L}\right)} .$$

3. MB 78.10

For a circuit consisting of R, L and C, all in parallel:

a) Find ω in terms of R, L, and C if the angle of Z is 45°. The total impedence for this system is

$$\frac{1}{Z} = \frac{1}{Z_R} + \frac{1}{Z_L} + \frac{1}{Z_C} . \tag{1}$$

In this case the simpliest way to find impedence is to directly plug in $Z_R = R$, $Z_L = i\omega L$, and $Z_C = 1/(i\omega C)$

$$\begin{aligned} \frac{1}{Z} &= \frac{1}{R} + \frac{1}{i\omega L} + \frac{1}{1/(i\omega C)} \\ &= \frac{1}{R} + i\left(\omega C - \frac{1}{\omega L}\right) . \end{aligned} \tag{2}$$

Next we invert Eq. (2) and divide to find

$$Z = \frac{1/R - i[\omega C - 1/(\omega L)]}{R^2 + [\omega C - 1/(\omega L)]^2} . \tag{3}$$

Now we recall that $\operatorname{Im}\{z\}/\operatorname{Re}\{z\} = \tan(\theta)$. For Eq. (2) this is

$$\frac{(1/(\omega L) - \omega C)}{1/R} = \tan \pi/4 = 1 . \tag{4}$$

We can write this relation in the form of a quadratic equation:

$$\omega^2 + \frac{1}{RC}\omega - \frac{1}{LC} = 0 . \tag{5}$$

Hence, the solution is

$$\boxed{\omega = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}} . \tag{6}$$

b) Find the resonant frequency ω for this circuit.

The resonance frequency occurs when the imaginary part of the impedence vanishes, so by examining Eq. (2), we see there is resonance at $\omega C - 1/(\omega L) = 0$. Solving this we get

$$\boxed{\omega^2 = \frac{1}{LC}} .$$

4. MB 79.12 Given

$$\left(\sum_{n=0}^{\infty} r^{2n} \cos n\theta\right)^2 + \left(\sum_{n=0}^{\infty} r^{2n} \sin n\theta\right)^2 , \tag{7}$$

show that this is equal to

$$\left|\sum_{n=0}^{\infty} r^{2n} e^{in\theta}\right|^2 , \tag{8}$$

assuming $|r| < 1$.

The first step is to recognize that

$$\begin{aligned} \mathrm{Re}\left\{\sum_{n=0}^{\infty} r^{2n} e^{in\theta}\right\} &= \sum_{n=0}^{\infty} r^{2n} \cos n\theta , \\ \mathrm{Im}\left\{\sum_{n=0}^{\infty} r^{2n} e^{in\theta}\right\} &= \sum_{n=0}^{\infty} r^{2n} \sin n\theta . \end{aligned} \tag{9}$$

Next we see that $S = \sum_{n=0}^{\infty} r^{2n} e^{in\theta}$ is a complex geometric series which converges to $S = 1/(1-z)$ with $z = r^2 e^{i\theta}$. Note that S is a complex number, so the absolute value of $|S|^2 = \mathrm{Re}\{S\}^2 + \mathrm{Im}\{S\}^2$. Now we can say that

$$\boxed{\left|\sum_{n=0}^{\infty} r^{2n} e^{in\theta}\right|^2 = \left(\sum_{n=0}^{\infty} r^{2n} \cos n\theta\right)^2 + \left(\sum_{n=0}^{\infty} r^{2n} \sin n\theta\right)^2} . \tag{10}$$

5. MB 88.7 Write the following as an augmented matrix and put it row echelon form to find out whether the given set of equations has exactly one solution, no solutions, or infinite set of solutions:

$$\begin{cases} 2x + 3y = 1 \\ x + 2y = 2 \\ x + 3y = 5 \end{cases}$$

The augmented matrix is

$$\left(\begin{array}{cc|c} 2 & 3 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 5 \end{array}\right) .$$

We begin by row reducing the first column:

$$\left(\begin{array}{cc|c} 2 & 3 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 5 \end{array}\right) \xrightarrow[R_2 \to R_2 - 1/2 R_1]{R_3 \to R_3 - 1/2 R_1} \left(\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & 1/2 & 3/2 \\ 0 & 3/2 & 9/2 \end{array}\right) ,$$

where $R_n \to R_n + aR_m$ means row n goes to row n plus a times row m for any constant a. Now we can see that R_2 is portional to R_3 such that

$$\left(\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & 1/2 & 3/2 \\ 0 & 3/2 & 9/2 \end{array}\right) \xrightarrow{R_3 \to R_3 - 3R_1} \left(\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & 1/2 & 3/2 \\ 0 & 0 & 0 \end{array}\right) .$$

This is called triangular form. To put this in row echelon form, we have to make the first non-zero element of each row equal to one:

$$\left(\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & 1/2 & 3/2 \\ 0 & 0 & 0 \end{array}\right) \xrightarrow[R_1 \to 1/2 R_1]{R_2 \to 2R_2} \left(\begin{array}{cc|c} 1 & 3/2 & 1/2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right) .$$

Now, we are in row echelon form. In order to put this in reduced row echlon form, we have to zero out all the elements above the first non-zero element of each row:

$$\left(\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & 1/2 & 3/2 \\ 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1 \to R_1 - 3/2 R_2} \left(\begin{array}{cc|c} 1 & 0 & -4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right) .$$

$\boxed{\text{We can immediately see that there is one solution with } x = -4 \text{ and } y = 3.}$

6. MB 88.8

   Write the following as an augmented matrix and put it row echelon form to find out whether the given set of equations has exactly one solution, no solutions, or infinite set of solutions:

$$\begin{cases} -x + y - z = 4 \\ x - y + 2z = 3 \\ 2x - 2y + 4z = 6 \end{cases}$$

   The augmented matrix is

$$\left(\begin{array}{ccc|c} -1 & 1 & -1 & 4 \\ 1 & -1 & 2 & 3 \\ 1 & -2 & 4 & 6 \end{array}\right) .$$

We start by noticing that R_2 is porporational to R_3 such that

$$\left(\begin{array}{ccc|c} -1 & 1 & -1 & 4 \\ 1 & -1 & 2 & 3 \\ 1 & -2 & 4 & 6 \end{array}\right) \xrightarrow{R_3 \to R_3 - 2R_2} \left(\begin{array}{ccc|c} -1 & 1 & -1 & 4 \\ 1 & -1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

We continue by adding the R_1 to R_2:

$$\left(\begin{array}{ccc|c} -1 & 1 & -1 & 4 \\ 1 & -1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_2 \to R_2 + R_1} \left(\begin{array}{ccc|c} -1 & 1 & -1 & 4 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

In order to put this in the row echelon form, we have to make all the elements above the first non-zero element of each row equal to zero.

$$\left(\begin{array}{ccc|c} -1 & 1 & -1 & 4 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[R_1 \to -1R_1]{R_2 \to R_2 + R_1} \left(\begin{array}{ccc|c} 1 & -1 & 0 & -11 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

$\boxed{\text{Hence, there are infinitely many solutions with } z = 7 \text{ and } y = x + 11.}$

7. MB 88.15

Find the rank of the follow matrix

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & 4 & 6 \\ 3 & 2 & 5 \end{pmatrix}.$$

The most direct way to find the rank of the matrix to put the matrix in row echelon form. We start by zeroing out the first elements of R_2 and R_3:

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & 4 & 6 \\ 3 & 2 & 5 \end{pmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & -1 & -1 \end{pmatrix}.$$

Next, we see that R_2 is porportional to R_3:

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & -1 & -1 \end{pmatrix} \xrightarrow{R_2 \to R_2 + 1/2R_1} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now, we set the pivot point equal to 1:

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2 \to 1/2R_1} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Finally, we zero out the element above the pivot point.

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 \to R_1 - R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The rank of a matrix is given by the number of linearly independent equations. $\boxed{\text{Hence, the rank is 2.}}$

8. MB 88.18 Find the rank of the follow matrix

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ -1 & -2 & -1 & 0 \\ 2 & 2 & 5 & 3 \\ 2 & 4 & 8 & 6 \end{pmatrix}.$$

The trick to determining the rank of the matrix is put it in reduced row echelon form. We start by zeroing out the frist column:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ -1 & -2 & -1 & 0 \\ 2 & 2 & 5 & 3 \\ 2 & 4 & 8 & 6 \end{pmatrix} \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 2R_1} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 2 & 3 & 3 \\ 2 & 4 & 6 & 6 \end{pmatrix} \xrightarrow{R_4 \to R_4 - 2R_1} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 2 & 3 & 3 \\ 0 & 4 & 6 & 6 \end{pmatrix}.$$

We can see that R_3 is porporational to R_4

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 2 & 3 & 3 \\ 0 & 4 & 6 & 6 \end{pmatrix} \xrightarrow{R_4 \to R_4 - 2R_3} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 2 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Next we zero out the coeffiencts below the first non-zero coefficent of the second row:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 2 & 3 & 3 \\ 0 & 4 & 6 & 6 \end{pmatrix} \xrightarrow{R_3 \to R_3 + R_1} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since all coeffiencent below the leading coeffiecent in third row are already zero and there are no non-zero coeffiencents in the 4th row, we start the next phrase of the algorithm. We set the leading coefficent of the third row to one and zero out all the element about the leading coeffienct:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow[R_1 \to R_1 - R_3]{R_3 \to 1/3 R_3} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Finally, we set the leading coeffiecent of the second row equal to one

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2 \to -1/2 R_2} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The rank of a matrix is equal to the number of linearly indepedent equations. $\boxed{\text{Hence, the rank of this matrix is 3.}}$

9. MB 95.4 Evaluate the determinate using the method described in MB 92. Ex 4:

$$D = \begin{vmatrix} -2 & 4 & 7 & 3 \\ 8 & 2 & -9 & 5 \\ -4 & 6 & 8 & 4 \\ 2 & -9 & 3 & 8 \end{vmatrix} .$$

We start by setting to zero all the element in the first column under the under the frist element.

$$\begin{vmatrix} -2 & 4 & 7 & 3 \\ 8 & 2 & -9 & 5 \\ -4 & 6 & 8 & 4 \\ 2 & -9 & 3 & 8 \end{vmatrix} \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 + 4R} \begin{vmatrix} -2 & 0 & 0 & -1 \\ 0 & 18 & 19 & 17 \\ 0 & -2 & -6 & -2 \\ 2 & -9 & 3 & 8 \end{vmatrix} \xrightarrow{R_2 \to R_2 + R_1} \begin{vmatrix} -2 & 0 & 0 & -1 \\ 0 & 18 & 19 & 17 \\ 0 & -2 & -6 & -2 \\ 0 & -5 & 10 & 11 \end{vmatrix}$$

Now, we do a Laplace development using the first column:

$$D = (-2) \begin{vmatrix} 18 & 19 & 17 \\ -2 & -6 & -2 \\ -5 & 10 & 11 \end{vmatrix}$$

Next, we zero out all but the first element of the 2nd row:

$$D = (-2) \begin{vmatrix} 18 & 19 & 17 \\ -2 & -6 & -2 \\ -5 & 10 & 11 \end{vmatrix} \xrightarrow[C_3 \to C_3 - C_1]{C_2 \to C_2 - 3C_1} \begin{vmatrix} 18 & 35 & -1 \\ -2 & 0 & 0 \\ -5 & 25 & 16 \end{vmatrix} .$$

where C_n is the nth column. Now we do a Laplace development of the 2nd row:

$$D = (-2)(-2)(-1) \begin{vmatrix} -35 & -1 \\ 25 & 16 \end{vmatrix}$$

We zero out one of the elements:

$$\begin{vmatrix} 35 & -1 \\ 25 & 16 \end{vmatrix} \xrightarrow{R_2 \to R_2 + 16R_1} \begin{pmatrix} 35 & -1 \\ -535 & 0 \end{pmatrix} .$$

Finally, we do one last Laplace development of the 1st column:

$$D = (-2)(-2)(-1)[0 - (-535)(-1)] = \boxed{2040} .$$

10. MB 95.7

Prove the following by appropriate manipulations using Facts 1 to 4:

$$\begin{vmatrix} 1 & a & bc \\ 1 & b & ac \\ 1 & c & ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (c-a)(b-a)(c-b) \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & a+b \\ 0 & 0 & 1 \end{vmatrix} = (c-a)(b-a)(c-b) .$$

There many ways to show these relations are equal. In any case we must do at least three separate evaluations:

a) Let's start by showing that

$$\begin{vmatrix} 1 & a & bc \\ 1 & b & ac \\ 1 & c & ab \end{vmatrix} = (c-a)(b-a)(c-b) \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & a+b \\ 0 & 0 & 1 \end{vmatrix} .$$

Use Fact 4 on the 1st column:

$$\begin{vmatrix} 1 & a & bc \\ 1 & b & ac \\ 1 & c & ab \end{vmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \begin{vmatrix} 1 & a & bc \\ 0 & b-a & (a-b)c \\ 0 & c-a & (a-c)b \end{vmatrix} .$$

Pull out $(b-a)$ from the R_2 and $(c-a)$ from the R_3 using Fact 1:

$$(b-a)(c-a) \begin{vmatrix} 1 & a & bc \\ 0 & 1 & -c \\ 0 & 1 & -b \end{vmatrix} .$$

Ignoring the factors in front for now, use Fact 4 on the 2nd column:

$$\begin{vmatrix} 1 & a & bc \\ 0 & 1 & -c \\ 0 & 1 & -b \end{vmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{vmatrix} 1 & a & bc \\ 0 & 1 & -c \\ 0 & 0 & c-b \end{vmatrix} .$$

Pull out $(c-b)$ from R_3 using Fact 1:

$$(c-a)(b-a)(c-a) \begin{vmatrix} 1 & a & bc \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{vmatrix} .$$

Ignoring the factors in front, use Fact 4 on the 3rd column:

$$\begin{vmatrix} 1 & a & bc \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{vmatrix} \xrightarrow[R_1 \to R_1 - (bc-a^2)R_3]{R_2 \to R_2 - (-c-a+b)R_3} \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & a+b \\ 0 & 0 & 1 \end{vmatrix} .$$

Hence,

$$\boxed{\begin{vmatrix} 1 & a & bc \\ 1 & b & ac \\ 1 & c & ab \end{vmatrix} = (c-a)(b-a)(c-b) \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & a+b \\ 0 & 0 & 1 \end{vmatrix}} .$$

b) Next let's show that

$$(c-a)(b-a)(c-b) \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & a+b \\ 0 & 0 & 1 \end{vmatrix} = (c-a)(b-a)(c-b) .$$

At this point we just need to evaluate the determinate:

$$\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & a+b \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & a+b \\ 0 & 1 \end{vmatrix} = [1-(a+b)0] = 1\ . \tag{11}$$

Hence,

$$\boxed{(c-a)(b-a)(c-b)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & a+b \\ 0 & 0 & 1 \end{vmatrix} = (c-a)(b-a)(c-b)}\ .$$

c) Finally, let's show that

$$(c-a)(b-a)(c-b)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & a+b \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}\ .$$

Start by using Fact 1 to pull $(c-a)$ into the 3rd row:

$$(b-a)(c-b)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & a+b \\ 0 & 0 & c-a \end{vmatrix}\ .$$

Next use Fact 4 to add R_2 to R_3:

$$(b-a)(c-b)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & a+b \\ 0 & 1 & c+b \end{vmatrix}\ .$$

Use Fact 1 to pull $(b-a)$ into R_2:

$$(c-b)\begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & 1 & c+b \end{vmatrix}\ .$$

Use Fact 4 to add R_1 to R_2:

$$(c-b)\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 0 & 1 & c+b \end{vmatrix}\ .$$

Use Fact 1 to pull $(c-b)$ into R_3:

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 0 & c-b & c^2-b^2 \end{vmatrix}\ .$$

Use Fact 4 to add R_2 to R_3:

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} .$$

Hence,

$$\boxed{(c-a)(b-a)(c-b)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & a+b \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}} .$$