Mathematical Methods of Physics 116A- Winter 2018

Physics 116A

Home Work # 5 Solutions
Posted on Feb 8, 2018
Due in Class Feb 15, 2018

§**Required Problems: Each problem has 10 points**

E.g. MB 19.16 means problem #16 on page 19 in the book by M. Boas, 3rd Edition.

1. MB 95.10 Show that a skew-symmetric matrix determinant of odd order is zero.

   Using the notation $\det(A)$ for the determinate of an $n \times n$ matrix A, we can write a very compact proof:

   $$\det(A) = \det\left(A^T\right) = \det(-A) = (-1)^n \det(A)\ .$$

   We use the following properties:

   a) $\det(A) = \det\left(A^T\right)$ (in general)
   b) $A^T = -A$ (for a skew symmetric matrix)
   c) $\det(-A) = (-1)^n \det(A)$ (The determinate of $-A$ is the same as negating each row of the $\det(A)$).

   $$\boxed{\text{Hence, for } n \text{ odd we have a contradition } \det(A) = -\det(A) \text{ unless } \det(A) = 0.}$$

2. MB 96.18

   Find the solution of z using Cramer's rule given the following set of equations:

   $$\begin{cases} (a-b)x - (a+b)y + 3b^2 z = 3ab \\ (a+2b)x - (a+2b)y - (3ab+3b^2)z = 3b^2 \\ bx + ay - (2b^2+a^2)z = 0 \end{cases}\ .$$

   First we need compute the determinate

   $$D = \begin{vmatrix} (a-b) & -(a-b) & 3b^2 \\ (a+2b) & -(a+2b) & -(3ab+3b^2) \\ b & a & (2b^2+a^2) \end{vmatrix}\ .$$

   Add C_2 to C_1:

   $$D = \begin{vmatrix} 0 & -(a-b) & 3b^2 \\ 0 & -(a+2b) & -(3ab+3b^2) \\ a+b & a & (2b^2+a^2) \end{vmatrix}\ .$$

Do a Laplace developement of C_1:

$$D = (a+b)\begin{vmatrix} -(a-b) & 3b^2 \\ -(a+2b) & -(3ab+3b^2) \end{vmatrix} .$$

Pull $3b$ out of C_2 and compute the 2 determinate:

$$D = 3b(a+b)\begin{vmatrix} -(a-b) & b \\ b & -(a+b) \end{vmatrix} = 3b(a+b)(a^2+ab+b^2) .$$

Next, we use Cramer's rule to find z,

$$z = \frac{1}{D}\begin{vmatrix} (a-b) & -(a-b) & 3ab \\ (a+2b) & -(a+2b) & 3b^2 \\ b & a & 0 \end{vmatrix}$$

where the determinate in the numerator is found by replacing the coefficents for z with the coefficents of the solution. Add C_1 to C_2:

$$\begin{vmatrix} 0 & -(a-b) & 3ab \\ 0 & -(a-2b) & 3b^2 \\ a+b & a & 0 \end{vmatrix} .$$

Do a Laplace developement of C_1:

$$(a+b)/D\begin{vmatrix} -(a-b) & 3ab \\ -(a+2b) & 3b^2 \end{vmatrix} .$$

Pull $3b$ out of C_2:

$$3b(a+b)/D\begin{vmatrix} -(a-b) & a \\ -(a+2b) & b \end{vmatrix} .$$

Pull $3b$ out of C_2 and compute the determinate:

$$3b(a+b)/D\begin{vmatrix} -(a-b) & a \\ -(a+2b) & b \end{vmatrix} = 3b(a+b)(a^2+ab+b^2)/D = 1 .$$

3. MB 112.20 Find the equation of a plane that passes through the points $P(0,1,1)$, $Q(2,1,3)$, and $R(4,2,1)$.

   For every plane there exist a vector normal to all vectors in the plane which we call $\vec{N}$. We can compute the normal vector by taking the cross product of any two vectors non-parallel vectors that lie in the plane. Note that the vectors that joins points P, Q, or R lies in the plane. Two such vectors are $\vec{PQ} = (2,0,2)$ and $\vec{PR} = (4,1,0)$. The cross product is

$$\overrightarrow{N} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 2 & 0 & 2 \\ 4 & 1 & 0 \end{vmatrix} = -2\hat{x} + 8\hat{y} + 2\hat{z} = (-2,8,2) .$$

If $\vec{r} - \vec{r}_0$ defines an arbitrary vector in the plane, then

$$\vec{N} \cdot (\vec{r} - \vec{r}_0) = 0$$

since the normal vector is perpendicular to any vector in the plane. So if we let $\vec{r_0} = \vec{PO} = (0, 1, 1)$ where $O(0, 0, 0)$ is the origin and $\vec{r} = (x, y, z)$ we have

$$\vec{N} \cdot (\vec{r} - \vec{r}_0) = -2(x - 0) + 8(y - 1) + 2(z - 1) = 0 \; .$$

Hence,

$$\boxed{x - 4y - z + 5 = 0} \; .$$

4. MB 112.22 Find the angle between the two planes

$$\begin{cases} 2x - y - z = 4 \\ 3x - 2y - 6z = 7 \end{cases} \; .$$

The angle between two planes is equivalent to the angle between the normal vectorw of the two planes. We can find the norm of the first plane, $\vec{N}_1 = (2, -1, -1)$, by reading off the coefficents. Similarly for the second equation, we find $\vec{N}_2 = (3, -2, -6)$. The dot product of the norms is defined at

$$\vec{N}_1 \cdot \vec{N}_2 = |\vec{N}_1||\vec{N}_2| \cos\theta \; ,$$

where $|\vec{N}_1| = \sqrt{6}$ and $|\vec{N}_2| = \sqrt{7}$. Thus

$$\boxed{\cos\theta = \frac{2}{\sqrt{6}}} \; .$$

5. MB 113.31 Find the distance from $P(-2, 4, 5)$ to the plane $2x + 6y - 3z = 10$.

We can find the distance from a point to a plane by defining the vector $\overrightarrow{QP}$ to be an arbitary point $Q(x, y, z)$ on the plane to the point P and projecting that on to the normalized normal vector of the plane. The normal vector of the plane $\overrightarrow{N} = (2, 6, -3)$ and $|\vec{N}| = 7$. The distance is defined at

$$|\hat{N} \cdot \overrightarrow{QP}| = \boxed{\frac{5}{7}}$$

6. MB 113.27 Find the plane that passes through $P(2, 3, -2)$ and is perpendicular to the planes

$$\begin{cases} 2x + 6y - 3z = 10 \\ 5x + 2y - z = 12 \end{cases} \; .$$

The orientation of a normal vector of a plane that is perpendicular to two other planes is defined by the cross product of the normal vectors of two other planes. In our case we let $\vec{A} = (2, 6, -3)$ be the normal vector of the first plane and $\vec{B} = (5, 2, -1)$ be the normal vector of the second plane. The cross product gives the normal vector of our plane

$$\vec{C} = \vec{A} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 2 & 6 & -3 \\ 5 & 2 & -1 \end{vmatrix} = 0\hat{x} - 13\hat{y} - 26\hat{z} = (0, -13, -26) \; .$$

The equation of a plane is given by

$$\vec{C} \cdot (\vec{r} - \vec{r}_0) = 0(x - 2) - 13(y - 3) - 26(z + 2) = 0 \; ,$$

where $\vec{r}_0 = (2, 3, -2)$. This simplifies to

$$\boxed{y + 2z + 1 = 0} \; .$$