Mathematical Methods of Physics 116A- Winter 2018

Physics 116A

Home Work # 6 Solutions
Posted on Feb 15, 2018
Due in Class Feb 22, 2018

§**Required Problems: Each problem has 10 points**

E.g. MB 19.16 means problem #16 on page 19 in the book by M. Boas, 3rd Edition.

1. MB 113.43

   Find the shortest distance between the two lines:

$$\frac{x-1}{2} = \frac{y+2}{3} = \frac{2z-1}{-1} = t$$

   and

$$\frac{x+2}{-1} = \frac{2-y}{2} = s\ , \quad z = 1/2\ .$$

   In parameteric form these lines are $\vec{r}_1 = (2t+1, 3t-2, 2t+1/2)$ and $\vec{r}_2 = (-s-2, -2s+2, 1/2)$. The direction of lines $\vec{r}_1$ and $\vec{r}_2$ are $\vec{n}_1 = (2,3,2)$ and $\vec{n}_2 = (-1,-2,0)$, respectively. We can immediately see that lines are not parallel. Notice that the cross product of n_1 and n_2 defines the normal vector of a plane to which both lines are parallel. To find the shortest distance between the two lines we must project an arbituary vector ($\overrightarrow{PQ}$) that connects lines $\vec{r}_1$ and $\vec{r}_2$ onto the normal vector of this plane. First we must compute the cross product:

$$\vec{N} = n_1 \times n_2 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 2 & 3 & 2 \\ -1 & -2 & 0 \end{vmatrix} = (4, -2, -1)\ .$$

   The normal vector is $\hat{N} = \vec{N}/|\vec{N}| = (4,-2,-1)/\sqrt{21}$. For simplicity we choose the points on $\vec{r}_1$ and $\vec{r}_2$ to be $P(1,-2,1/2)$ and $Q(-2,2,1/2)$, respectively, by setting $t = s = 0$ such that $\overrightarrow{PQ} = (3,-3,3)$. Hence, the shortest distance between two lines is

$$\boxed{|\hat{N}\cdot\overrightarrow{PQ}| = \frac{20}{\sqrt{21}}}\ .$$

2. MB 113.45 Consider the line in symmetric form

$$\frac{(x-3)}{2} = \frac{(y+1)}{-2} = z - 1 = t\ .$$

a) Express the line in parametric form $\vec{r} = \vec{r}_0 + At$:

$$\boxed{\vec{r} = (3, -1, 1) + (2, -2, 1)t} \ ,$$

where $\vec{A} = (2, -2, 1)$ and $\vec{r}_0 = (3, -1, 1)$.

b) Find the distance of closest apporach from the origin.

Let $O(0, 0, 0)$ be the point at the origin, $R(2t+3, -2t-1, t+1)$ be any point along the line, $\hat{u} = \vec{A}/|\vec{A}| = (2, -2, 1)/3$ be the unit vector along the line, and P be the point of closest approach. The distance from O to P is given by $d = |\overrightarrow{OR} \times \vec{u}| = |\overrightarrow{OP}|$. For simplicity we set $t = 0$ such that $\overrightarrow{OR} = (3, -1, 1)$. Now we compute:

$$d = |\overrightarrow{OQ} \times \hat{u}| = \frac{1}{3}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 3 & -1 & 1 \\ 2 & -2 & 1 \end{vmatrix} = \frac{1}{3}|(1, -1, -4)| \ .$$

Hence, $\boxed{d = \sqrt{2}}$.

c) Show that $t = -(\vec{r}_0 \cdot A)/|\vec{A}|^2$ is the time of closest approach from the origin.

Start with the parametric equation of a line:

$$\vec{r} = \vec{r}_0 + \vec{A}t$$

Next we take the dot product of both sides with respect to $\vec{A}$:

$$\vec{A} \cdot \vec{r} = \vec{A} \cdot \vec{r}_0 + \vec{A} \cdot \vec{A}t \ ,$$

where $\vec{A} \cdot \vec{A} = |\vec{A}|^2$.

Notice that at the distance of closest approach $\vec{A} \cdot \vec{r} = 0$ since they are perpedicular. Now we can solve for t to show that

$$\boxed{t = -\frac{\vec{A} \cdot \vec{r}_0}{|A|}} \ .$$

Also using this relation we find that the time of closest approach is $\boxed{t = -1}$ and the point of closest approach is $\boxed{P(1, 1, 0)}$.

3. MB 122.3 Consider the matricies

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 3 & -1 & 0 \\ 0 & 5 & 1 \end{pmatrix} \ , \quad B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 3 & -1 & 0 \end{pmatrix} \ .$$

a) Compute the matrices:

$$AB = \begin{pmatrix} 7 & -1 & 0 \\ 3 & 1 & -1 \\ 3 & 9 & 5 \end{pmatrix}, \quad BA = \begin{pmatrix} 4 & -1 & 2 \\ 6 & 3 & 1 \\ 0 & 1 & 6 \end{pmatrix},$$

$$A{+}B = \begin{pmatrix} 2 & 1 & 2 \\ 3 & 1 & 1 \\ 3 & 4 & 1 \end{pmatrix}, \quad A{-}B = \begin{pmatrix} 7 & -1 & 0 \\ 3 & 1 & -1 \\ 3 & 9 & 5 \end{pmatrix}, \quad A^2 = \begin{pmatrix} 1 & 10 & 4 \\ 0 & 1 & 6 \\ 15 & 0 & 1 \end{pmatrix},$$

$$B^2 = \begin{pmatrix} 1 & 3 & 1 \\ 3 & 3 & 2 \\ 3 & 1 & -1 \end{pmatrix}, \quad 5A = \begin{pmatrix} 5 & 0 & 10 \\ 15 & -5 & 0 \\ 0 & 25 & 5 \end{pmatrix}, \quad 3B = \begin{pmatrix} 3 & 3 & 0 \\ 0 & 6 & 3 \\ 9 & -3 & 0 \end{pmatrix}.$$

b) Show that $AB \neq BA$:

$$AB = \begin{pmatrix} 7 & -1 & 0 \\ 3 & 1 & -1 \\ 3 & 9 & 5 \end{pmatrix} \neq \begin{pmatrix} 3 & 3 & 0 \\ 0 & 6 & 3 \\ 9 & -3 & 0 \end{pmatrix} = BA\,.$$

c) Show that $(A+B)(A-B) \neq (A-B)(A+B) \neq A^2 - B^2$:

$$(A+B)(A-B) = \begin{pmatrix} 3 & 7 & 1 \\ -6 & -4 & 2 \\ 15 & 7 & 1 \end{pmatrix} \neq (A-B)(A+B) = \begin{pmatrix} -3 & 7 & 5 \\ 0 & 0 & 6 \\ 9 & -9 & 3 \end{pmatrix}$$

$$\neq A^2 - B^2 = \begin{pmatrix} 0 & 7 & 3 \\ -3 & -2 & 4 \\ 12 & -1 & 2 \end{pmatrix}.$$

d) Show that $\det(AB) = \det(BA) = \det(A)\det(B)$:

$$\det(AB) = \begin{vmatrix} 7 & -1 & 0 \\ 3 & 1 & -1 \\ 3 & 9 & 5 \end{vmatrix} = 116\,, \quad \det(BA) = \begin{vmatrix} 3 & 3 & 0 \\ 0 & 6 & 3 \\ 9 & -3 & 0 \end{vmatrix} = 116\,,$$

$$\det(A) = \begin{vmatrix} 1 & 0 & 2 \\ 3 & -1 & 0 \\ 0 & 5 & 1 \end{vmatrix} = 29\,, \quad \det(B) \begin{vmatrix} 1 & 0 & 2 \\ 3 & -1 & 0 \\ 0 & 5 & 1 \end{vmatrix} = 4\,.$$

Hence, $\det(AB) = \det(BA) = \det(A)\det(B) = 116$.

e) Show that $\det(A+B) \neq \det(A) + \det(B)$:

$$\det(A+B) = \begin{vmatrix} 2 & 1 & 2 \\ 3 & 1 & 1 \\ 3 & 4 & 1 \end{vmatrix} = 12 \, ,$$

Therefore, $\det(A+B) = 12 \neq \det(A) + \det(B) = 33$.

f) Show that $\det(5A) \neq 5\det(A)$ and $\det(3B) \neq 3\det(B)$:

$$\det(5A) = \begin{vmatrix} 7 & -1 & 0 \\ 3 & 1 & -1 \\ 3 & 9 & 5 \end{vmatrix} = 3625 \, , \quad \det(3B) = \begin{vmatrix} 3 & 3 & 0 \\ 0 & 6 & 3 \\ 9 & -3 & 0 \end{vmatrix} = 108$$

Observe that $\det(5A) = 3625 \neq 145 = 5\det(A)$ and $\det(3B) = 108 \neq 12 = \det(B)$.

g) Show that $\det(5A) = 5^3 \det(A)$ and $\det(3B) = 3^3 \det(B)$:

Clearly $\det(5A) = 3625 = 5^3 \det(A)$ and $\det(3B) = 108 = 3^3 \det(B)$.

4. MB 122.5

Compute the product of the matrices with its transpose, i.e. AA^T and A^TA:

$$A = \begin{pmatrix} 2 & 3 & 1 & -4 \\ 2 & 1 & 0 & 5 \end{pmatrix} \, , \quad B = \begin{pmatrix} 2 & 4 \\ 1 & -1 \\ 3 & -1 \end{pmatrix} \quad C = \begin{pmatrix} 2 & 1 & 3 \\ 4 & -1 & -2 \\ -1 & 0 & 0 \end{pmatrix} \, .$$

Do matrix multiplication to find:

$$AA^T = \begin{pmatrix} 30 & -13 \\ -13 & 30 \end{pmatrix} \, , \quad A^TA = \begin{pmatrix} 8 & 8 & 2 & 2 \\ 8 & 10 & 3 & -7 \\ 2 & 3 & 1 & -4 \\ 2 & 1 & -4 & 41 \end{pmatrix}$$

$$BB^T = \begin{pmatrix} 20 & -2 & 2 \\ -2 & 2 & 4 \\ 2 & 4 & 10 \end{pmatrix} \, , \quad B^TB = \begin{pmatrix} 14 & 4 \\ 4 & 18 \end{pmatrix}$$

$$CC^T = \begin{pmatrix} 14 & 1 & 1 \\ 1 & 21 & -6 \\ 1 & -6 & 2 \end{pmatrix} \, , \quad C^TC = \begin{pmatrix} 21 & -2 & -3 \\ -2 & 2 & 5 \\ -3 & 5 & 14 \end{pmatrix} \, .$$

5. MB 122.6 The Pauli spin matrices in quantum mechanics are:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \, , \quad B = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \, .$$

Show that $A^2 = B^2 = C^2 = I$.

The best way to compute this is to just matrix multiply:

$$A^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$B^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$C^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

First we'll compute AB, BA, BC, CB, AC, and CA:

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$BA = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$BC = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$CB = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

$$AC = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$CA = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \, .$$

By examining equations are above, we can see $AB = -BA$, $BC = -CB$, $AC = -CA$ all anti-communte. We can also see that commutation relation gives

$$AB - BA = \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix} = 2iC \, ,$$

$$BC - CB = \begin{pmatrix} 0 & 2i \\ 2i & 0 \end{pmatrix} = 2iA \, ,$$

$$CA - AC = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} = 2iB \, .$$

6. MB 122.8

   Show, by multiplying the matrices, that the following equation respresents an ellipse:

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 5 & -7 \\ 7 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 30 \ .$$

   Matrix multiple the LHS of the equation:

$$\begin{aligned} \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 5 & -7 \\ 7 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= (5x - 7y)x + (7x + 3y)y \\ &= 5x^2 + 3y^2 = 30 \ . \end{aligned}$$

   To put the expression in the form of an equation of an ellipse, we divide both sides of the equation by 30 now:

$$\frac{x^2}{6} + \frac{y^2}{10} = 1 \ ,$$

   where the ellipse is center at point $P(0,0)$ and has a semi-major axis of $a = \sqrt{10}$ and semi-minor axis of $b = \sqrt{6}$.

7. MB 123.30

   For the Pauli spin matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ ,$$

   find the matrices $\sin kA$, $\cos kA$, e^{kA}, and e^{ikA}.

   a) We can treat the matrix just like a variable and expand $\sin kA$ as a taylor series:

$$\sin kA = \sum_{n=0}^{\infty} \frac{(-1)^n (kA)^{2n+1}}{(2n+1)!}$$

   Notice that $A^{2n+1} = A$. Hence,

$$\boxed{A \sum_{n=0}^{\infty} \frac{(-1)^n k^{2n+1}}{(2n+1)!} = A \sin k} \ .$$

   b) Expand $\cos kA$ as a taylor series:

$$\cos kA = \sum_{n=0}^{\infty} \frac{(-1)^n (kA)^{2n}}{(2n)!}$$

   Notice that $A^{2n} = I$ where I is a 2 by 2 idenity matrix. Hence,

$$\boxed{I \sum_{n=0}^{\infty} \frac{(-1)^n k^{2n}}{(2n)!} = I \cos k} \ .$$

c) Expand e^{kA} as a taylor series:

$$e^{kA} = \sum_{n=0}^{\infty} \frac{(kA)^n}{n!}$$

Next separate the series into odd terms and even terms:

$$e^{kA} = \sum_{n=0}^{\infty} \frac{(kA)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(kA)^{2n+1}}{(2n+1)!} \ .$$

Now we can pull out the $A^{2n+1} = A$ and $A^{2n} = I$ from the series:

$$e^{kA} = I \sum_{n=0}^{\infty} \frac{(k)^{2n}}{(2n)!} + A \sum_{n=0}^{\infty} \frac{(k)^{2n+1}}{(2n+1)!}$$

Finally, we can see that these series are hyperoblic cosine and sine, respectively:

$$\boxed{e^{kA} = I \cosh k + A \sinh k}$$

d) Using Euler formula $e^{ikA} = \cos(kA) + i \sin(kA)$. Using the results of part a) and b) together with Euler's formula we find

$$\boxed{e^{ikA} = I \cos k + iA \sin k)} \ .$$

8. MB 123.32 For the Pauli spin matrix B:

$$B = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ .$$

a) Find $e^{i\theta B}$ and show that it is a rotation matrix.
First we expand $e^{i\theta B}$ using Euler formula:

$$e^{i\theta B} = \cos(\theta B) + \sin(\theta B)$$

. Next we expand $\sin(\theta B)$ and $\cos(\theta B)$ as a Taylor series:

$$e^{i\theta B} = \sum_{n=0}^{\infty} \frac{(-1)^n (\theta B)^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n (\theta B)^{2n+1}}{(2n+1)!}$$

Notice that $B^{2n+1} = B$ and $B^{2n} = I$ such that

$$e^{i\theta B} = I \sum_{n=0}^{\infty} \frac{(-1)^n (\theta)^{2n}}{(2n)!} + iB \sum_{n=0}^{\infty} \frac{(-1)^n (\theta)^{2n+1}}{(2n+1)!} \ .$$

Collapsing the series we get

$$e^{i\theta B} = I \cos\theta + iB \sin\theta \ .$$

In matrix form this is

$$\boxed{B = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}} \ ,$$

which is a clockwise rotation matrix.

b) Find $e^{-i\theta B}$ and show that it is a rotation matrix.

Following simily procedure as part a) we find that

$$e^{i\theta B} = I\cos\theta - iB\sin\theta \ .$$

In matrix form this is

$$\boxed{B = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}} \ ,$$

which is a counter-clockwise rotation matrix.

9. Are the following linear vector functions?

a) MB 130.5 $\vec{F}(\vec{r}) = \vec{A} \times \vec{r}$, where $\vec{A}$ is a vector.

Check that the function is linear under addition

$$\begin{aligned} \vec{F}(\vec{r}_1 + \vec{r}_2) &= \vec{A} \times (\vec{r}_1 + \vec{r}_2) \\ &= \vec{A} \times \vec{r}_1 + \vec{A} \times \vec{r}_2 \\ &= \vec{F}(\vec{r}_1) + \vec{F}(\vec{r}_2) \ , \end{aligned}$$

and check that the function is linear under scaler multiplication

$$\begin{aligned} \vec{F}(a\vec{r}) &= \vec{A} \times (a\vec{r}) \\ &= a\vec{A} \times \vec{r} \\ &= a\vec{F}(\vec{r}) \ . \end{aligned} \tag{1}$$

$\boxed{\text{Since it satifies both conditions, it is a linear function.}}$

b) MB 130.6 $\vec{F}(\vec{r}) = \vec{r} + \vec{A}$, where $\vec{A}$ is a vector.

Check that the function is linear under addition

$$\begin{aligned} \vec{F}(\vec{r}_1 + \vec{r}_2) &= \vec{r}_1 + \vec{r}_2 + \vec{A} \\ &\neq \vec{F}(\vec{r}_1) + \vec{F}\vec{r}_2) \end{aligned} \tag{2}$$

$\boxed{\text{Since it not linear under addition, it is not a linear fucntion.}}$