Mathematical Methods of Physics 116A- Winter 2018

Physics 116A

Home Work # 7 Solutions
Posted on Feb 22, 2018
Due in Class Mar 1, 2018

§**Required Problems: Each problem has 10 points**

E.g. MB 19.16 means problem #16 on page 19 in the book by M. Boas, 3rd Edition.

1. MB 131.20 Verify equation

$$\begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} .$$

The postion vector in the new coordinate system is $\mathbf{r}' = X\ \mathbf{i}' + Y\ \mathbf{j}'$. If we take the dot product of $\mathbf{r}' = X\mathbf{i}' + Y\mathbf{j}'$ with respect to the basis vectors of the new coordinate system, $\mathbf{i}'$ and $\mathbf{j}'$; we can find the matrix elements of the transformation matrix from old coordinates to new coordinates:

$$X' = \mathbf{i}' \cdot \mathbf{r}' = X\ \mathbf{i}' \cdot \mathbf{i}' + Y\ \mathbf{i}' \cdot \mathbf{j}'$$

where $\mathbf{i}' \cdot \mathbf{i}' = \cos\theta$ and $\mathbf{i}' \cdot \mathbf{j}' = \cos(\pi/2 - \theta) = \sin\theta$ and

$$Y' = \mathbf{j}' \cdot \mathbf{r}' = X\ \mathbf{j}' \cdot \mathbf{i}' + Y\ \mathbf{j}' \cdot \mathbf{j}'$$

where $\mathbf{j}' \cdot \mathbf{i}' = \cos(\theta + \pi/2) = -\sin\theta$ and $\mathbf{j}' \cdot \mathbf{j}' = \cos\theta$. Writing these two equations in matrix form we get

$$\boxed{\begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}} .$$

2. 131.29 Construct the matrix corresponding to a rotation of 90° about the y axis together with a reflection through the $x - z$ plane.

Note that a 90° rotation about the y axis cooresponds to a clockwise rotation. This is a matter of convention where we say that θ increases as we rotate a vector from the x axis to the z axis. The clockwise rotation matrix about the y axis is

$$R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

and a reflection about the $z - x$ plane is given by the matrix

$$P_{x-z} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We order the sequence of matrix transformations from right to left and then matrix multiply:

$$P_{x-z}R_y(90^\circ) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \boxed{\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}} .$$

3. MB 132.34

For the following matrix,

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

find its determinate to see whether it produces a reflection or a rotation. If it is a reflection, find the reflecting plane and then the rotation (if any) about the normal of the reflecting plane.

The determinate is

$$\det(A) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{vmatrix} = -1 .$$

Because the determinate is -1, we know this is an improper rotation: a rotation followed by a reflection or vice versa. In this case, first we reflect and then we rotate about $\mathbf{n}$, the normal vector to a some plane, i.e. $A = R_{\mathbf{n}}P_{\mathbf{n}}$. A vector $\mathbf{v}$ parallel to the normal of the reflecting plane satisife the relations $A\mathbf{v} = -\mathbf{v}$. If $\mathbf{v} = (a, b, c)$ then we solve for the $\mathbf{v}$ by examining the relations $a = -a$, $-c = -b$, and $-b = -c$ which tells us that $a = 0$ and $b = c$. This vector normalized is

$$\boxed{\frac{1}{\sqrt{2}}(0, 1, 1)} .$$

Hence, the reflecting plane is $\boxed{y + z = 0}$. The reflection transformation in matrix form is

$$P_{\mathbf{n}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} ,$$

so clearly, no further rotations is necessary. We can find this matrix explicitly by taking the matrix for a known reflection, $P_{\mathbf{n}}$, and use some rotation matrix, $R_{\mathbf{r}}(\theta)$, to rotate the normal vector of the plane into the new orientation, $P_{\mathbf{n}'}$. The transformation rule for this is $P_{\mathbf{n}'} = R_{\mathbf{r}}(\theta) P_{\mathbf{n}} R_{\mathbf{r}}^{-1}(\theta)$.

4. 135.2

   Find out whether the give vector are dependent or independent:

$$(1, -2, 3), (1, 1, 1), (-2, 1, -4), (3, 0, 5)$$

   if they are dependent find a linearly independent subset and write each given vector as a linear combination of the independent vectors.

$$\begin{pmatrix} 1 & -2 & 3 \\ 1 & 1 & 1 \\ -2 & 1 & -4 \\ 3 & 0 & 5 \end{pmatrix} \xrightarrow[R_3 \to R_3 + 2R_1]{R_2 \to R_2 - R_1} \begin{pmatrix} 1 & -2 & 3 \\ 0 & 3 & -2 \\ 0 & -3 & 2 \\ 3 & 0 & 5 \end{pmatrix} \xrightarrow{R_4 \to R_4 - R_1} \begin{pmatrix} 1 & -3 & 3 \\ 0 & 3 & -2 \\ 0 & -3 & 2 \\ 0 & 6 & -4 \end{pmatrix}$$

   Notice that R_3 and R_4 are porporational to R_2. Hence, they are linearly dependent, so we can reduce those bottom two row to zero:

$$\begin{pmatrix} 1 & -2 & 3 \\ 0 & 3 & -2 \\ 0 & -3 & 2 \\ 3 & 0 & 5 \end{pmatrix} \xrightarrow[R_3 \to R_3 + R_2]{R_4 \to R_4 - 2R_1} \begin{pmatrix} 1 & -2 & 3 \\ 0 & 3 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

   Next, we normalize the pivot element (boxed below) of R_2:

$$\begin{pmatrix} 1 & -2 & 3 \\ 0 & \boxed{3} & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2 \to 1/3 R_2} \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & -2/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

   Finally, we row reduce the element above the pivot element:

$$\begin{pmatrix} 1 & -2 & 3 \\ 0 & \boxed{1} & -2/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 \to R_1 + 2R_2} \begin{pmatrix} 1 & 0 & 5/3 \\ 0 & 1 & -2/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

   The subset of linearly indpendent vectors are $\boxed{\mathbf{v}_1 = (1, 0, 5/9)}$ and $\boxed{\mathbf{v}_2 = (0, 1, -2/3)}$ and the expand of the these vector in term of $\mathbf{v}_1$ and $\mathbf{v}_2$ are

$$\begin{aligned} (1, -2, 3) &= \mathbf{v}_1 - 2\mathbf{v}_2 \\ (1, 1, 1) &= \mathbf{v}_1 + \mathbf{v}_2 \\ (1, -2, 3) &= -2\mathbf{v}_1 + \mathbf{v}_2 \\ (1, 1, 1) &= 3\mathbf{v}_1 \end{aligned}.$$

5. 136.5 Show that any vector $\mathbf{V}$ in a plane can be written as a linear combination of two non-parallel vectors $\mathbf{A}$ and $\mathbf{B}$ in the plance, that is, find a and b so that $\mathbf{v} = a\mathbf{A} + b\mathbf{B}$.

   We start by taking the cross product of the vector $\mathbf{V}$ with respect to the basis vectors:

$$\begin{aligned} \mathbf{A} \times \mathbf{V} &= a\mathbf{A} \times \mathbf{A} + b\mathbf{A} \times \mathbf{B} \\ \mathbf{B} \times \mathbf{V} &= a\mathbf{B} \times \mathbf{A} + b\mathbf{B} \times \mathbf{B} \end{aligned} \tag{1}$$

   Recall that $\mathbf{A} \times \mathbf{A} = 0$ and $\mathbf{B} \times \mathbf{B} = 0$. Next, we take the projection of $(\mathbf{A} \times \mathbf{V})$ and $(\mathbf{B} \times \mathbf{V})$ onto the normal vector, $\mathbf{n} = (\mathbf{A} \times \mathbf{B})/|\mathbf{A} \times \mathbf{B}|$:

$$\begin{aligned} (\mathbf{A} \times \mathbf{V}) \cdot \mathbf{n} &= b(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{n} \\ (\mathbf{B} \times \mathbf{V}) \cdot \mathbf{n} &= a(\mathbf{B} \times \mathbf{A}) \cdot \mathbf{n} \; . \end{aligned} \tag{2}$$

   Finally, we solve for a and b:

$$\boxed{a = \frac{(\mathbf{B} \times \mathbf{V}) \cdot \mathbf{n}}{(\mathbf{B} \times \mathbf{A}) \cdot \mathbf{n}} \; , \qquad b = \frac{(\mathbf{A} \times \mathbf{V}) \cdot \mathbf{n}}{(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{n}}} \; .$$

   Notice that $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{n}$ gives the area of parallelogram formed by $\mathbf{A}$ and $\mathbf{B}$.

6. 137.22 Find a condition for three lines in a plane to intersect at one point.

   We can write the equation of the three lines as

$$\begin{cases} a_{11}x + a_{12}y = b_1 \\ a_{21}x + a_{22}y = b_2 \\ a_{31}x + a_{32}y = b_3 \end{cases}$$

   where a_{ij} and b_i are constants. The row vectors must be linearly independent. **In other words, there must exist a single solution to the augmented matrix.**

7. 137.25 Find the eigenvalues and the cooresponding eigenvectors for the following set of equations:

$$\begin{cases} -(1+\lambda)x + y + 3z &= 0 \\ x + (2-\lambda)y &= 0 \\ 3x + (2-\lambda)z &= 0 \end{cases} \; .$$

   In matrix form this is

$$A - \lambda I = \begin{pmatrix} -(1+\lambda) & 1 & 3 \\ 1 & (2-\lambda) & 0 \\ 3 & 0 & (2-\lambda) \end{pmatrix} \; , \tag{3}$$

where I is the idenity matrix. We can find the eigenvalues of the matrix by evaluating $\det(A - \lambda I) = 0$ and solving for λ. Evaluating the matrix using any method, we find the determinate is

$$\det(A - \lambda I) = -\lambda^3 + 3\lambda^2 + 10\lambda + 3\lambda - 24 = 0 \,.$$

Next, we factor to find that

$$(\lambda - 4)(\lambda - 2)(\lambda + 3) = 0 \,.$$

Hence the eigenvalues are $\boxed{\mathbf{r} = (4, 2, -3)}$. The corresponding eigenvectors are found by plugging λ into $A - \lambda I = \mathbf{0}$ and solving the augmented matrix. We find for $\lambda = 4$: $\boxed{\mathbf{v} = (2y, y, 3y)}$, $\lambda = 2$: $\boxed{\mathbf{v} = (0, -3z, z)}$ and $\lambda = -3$: $\boxed{\mathbf{v} = (-5y, y, 3y)}$.

8. 141.3 Find the transpose, the inverse, the complex conjugate of

$$A = \begin{pmatrix} 1 & 0 & 5i \\ -2i & 2 & 0 \\ 1 & 1+i & 0 \end{pmatrix} \,.$$

and verify that $AA^{-1} = A^{-1}A = I$, where I is a unit matrix.

The transpose is

$$\boxed{A^T = \begin{pmatrix} 1 & -2i & 1 \\ 0 & 2 & 1+i \\ 5i & 0 & 0 \end{pmatrix}} \,.$$

Another way to find the inverse is by row reducting this augmented matrix:

$$A^T = \left(\begin{array}{ccc|ccc} 1 & 0 & 5i & 1 & 0 & 0 \\ -2i & 2 & 0 & 0 & 1 & 0 \\ 1 & 1+i & 0 & 0 & 0 & 1 \end{array}\right) \,.$$

We start by row reducing the elements below the pivot element (boxed) of the first row:

$$A^T = \left(\begin{array}{ccc|ccc} \boxed{1} & 0 & 5i & 1 & 0 & 0 \\ -2i & 2 & 0 & 0 & 1 & 0 \\ 1 & 1+i & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 + 2iR_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 5i & 1 & 0 & 0 \\ 0 & 2 & -10 & 2i & 1 & 0 \\ 0 & 1+i & -5i & -1 & 0 & 1 \end{array}\right)$$

Next, we row reduce the elements below the pivot element of the second row:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 5i & 1 & 0 & 0 \\ 0 & \boxed{2} & -10 & 2i & 1 & 0 \\ 0 & 1+i & -5i & -1 & 0 & 1 \end{array}\right) \xrightarrow{R_3 \to R_3 - (1+i)/2R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 5i & 1 & 0 & 0 \\ 0 & 2 & -10 & 2i & 1 & 0 \\ 0 & 0 & 5 & -i & -i/2 & 1 \end{array}\right)$$

Now, we normalize the pivot element of the third row:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 5i & 1 & 0 & 0 \\ 0 & 2 & -10 & 2i & 1 & 0 \\ 0 & 0 & \boxed{5} & -i & -i/2 & 1 \end{array}\right) \xrightarrow{R_3 \to R_3 - (1+i)/2 R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 5i & 1 & 0 & 0 \\ 0 & 2 & -10 & 2i & 1 & 0 \\ 0 & 0 & 1 & -i/5 & -i/10 & 1/5 \end{array}\right)$$

Next, we row reduce the elements above the pivot element of the third row:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 5i & 1 & 0 & 0 \\ 0 & 2 & -10 & 2i & 1 & 0 \\ 0 & 0 & \boxed{1} & -i/5 & -i/10 & 1/5 \end{array}\right) \xrightarrow[R_1 \to R_1 - 5iR_3]{R_2 \to R_2 + 10R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1/2 + i/2 & -i \\ 0 & 2 & 0 & 0 & -i & 2 \\ 0 & 0 & 1 & -i/5 & -i/10 & 1/5 \end{array}\right)$$

Finally, we normalize the pivot element of the second row:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1/2 + i/2 & -i \\ 0 & \boxed{2} & 0 & 0 & -i & 2 \\ 0 & 0 & 1 & -i/5 & -i/10 & 1/5 \end{array}\right) \xrightarrow[R_1 \to R_1 - 5iR_3]{R_2 \to R_2 + 10R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1/2 + i/2 & -i \\ 0 & 1 & 0 & 0 & -i/2 & 1 \\ 0 & 0 & 1 & -i/5 & -i/10 & 1/5 \end{array}\right)$$

So the inverse matrix is

$$\boxed{A^{-1} = \begin{pmatrix} 0 & -1/2 + i/2 & -i \\ 0 & -i/2 & 1 \\ -i/5 & -i/10 & 1/5 \end{pmatrix}}.$$

Now we can matrix multiply to show that $A^{-1}A = AA^{-1} = I$:

$$AA^{-1} = \begin{pmatrix} 1 & 0 & 5i \\ -2i & 2 & 0 \\ 1 & 1+i & 0 \end{pmatrix} \begin{pmatrix} 0 & -1/2 + i/2 & -i \\ 0 & -i/2 & 1 \\ -i/5 & -i/10 & 1/5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A^{-1}A = \begin{pmatrix} 0 & -1/2 + i/2 & -i \\ 0 & -i/2 & 1 \\ -i/5 & -i/10 & 1/5 \end{pmatrix} \begin{pmatrix} 1 & 0 & 5i \\ -2i & 2 & 0 \\ 1 & 1+i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Hence, $\boxed{A^{-1}A = AA^{-1} = I}$.

The complex conjugate

$$\boxed{A^* = \begin{pmatrix} 1 & 0 & -5i \\ 2i & 2 & 0 \\ 1 & 1-i & 0 \end{pmatrix}}.$$

The complex conjugate

$$\boxed{(A^*)^T = \begin{pmatrix} 1 & 2i & 1 \\ 0 & 2 & 1-i \\ -5i & 0 & 0 \end{pmatrix}}.$$

9. 141.13 Show that the following matrix is a unitary matrix.

$$U = \begin{pmatrix} (1+i\sqrt{3})/4 & \frac{\sqrt{3}}{2\sqrt{2}}(1+i) \\ \frac{-\sqrt{3}}{2\sqrt{2}}(1+i) & (\sqrt{3}+i)/4 \end{pmatrix} .$$

For a unitary matrix $UU^\dagger = U^\dagger U = I$ where $\dagger$ indicates we take the complex transpose of the matrix and I is the identity matrix.

$$U^\dagger = \begin{pmatrix} (1-i\sqrt{3})/4 & \frac{-\sqrt{3}}{2\sqrt{2}}(1-i) \\ \frac{\sqrt{3}}{2\sqrt{2}}(1-i) & (\sqrt{3}-i)/4 \end{pmatrix} .$$

Now matrix multiply to show that

$$\boxed{UU^\dagger = \begin{pmatrix} (1+i\sqrt{3})/4 & \frac{\sqrt{3}}{2\sqrt{2}}(1+i) \\ \frac{-\sqrt{3}}{2\sqrt{2}}(1+i) & (\sqrt{3}+i)/4 \end{pmatrix} \begin{pmatrix} (1-i\sqrt{3})/4 & \frac{-\sqrt{3}}{2\sqrt{2}}(1-i) \\ \frac{\sqrt{3}}{2\sqrt{2}}(1-i) & (\sqrt{3}-i)/4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} .$$

10. 142.22 Show that a unitary matrix is a normal matrix, that is, it commutes with its transpose conjugate. Also show that orthogonal, symmetric, anti-symetric, Hermitian, and anti-Hermitian matrices are normal.

The complex tranpose of a unitary matrix is $(U_{ij})^\dagger = U^*_{ji} = (U^{-1})_{ij}$. The commutator is

$$[U, U^\dagger] = UU^\dagger - U^\dagger U = 0 ,$$

where 0 is a zero matrix. In index notation this is

$$\begin{aligned} ([U, U^\dagger])_{ij} &= \sum_k U_{ik} U^*_{kj} - \sum_k U^*_{ik} U_{kj} \\ &= \sum_k U_{ik} (U^{-1})_{kj} - \sum_k (U^{-1})_{ik} U_{kj} \\ &= \delta_{ij} - \delta_{ij} = 0 . \end{aligned}$$

Hence, the matrix is normal. Similarly, the complex tranpose of an orthogal matrix is $O^\dagger_{ij} = (O^{-1})_{ij}$, a symmetric matrix $S^\dagger_{ij} = S_{ji}$, an anti-symmetric matrix $A^\dagger_{ij} = -A_{ij}$, a Hermitian matrix $H^\dagger_{ij} = H_{ji}$, and an anti-Hermitian $G^\dagger_{ij} = -G_{ji}$. In all case,

$$([M, M^\dagger])_{ij} = \sum_k M_{ik} M_{kj} - \sum_k M_{ik} M_{kj} = 0 .$$

Hence, all these matrices are normal.