Mathematical Methods of Physics 116A- Winter 2018

Physics 116A

Home Work # 8 Solutions
Posted on Mar 1, 2018
Due in Class Mar 8, 2018

§**Required Problems: Each problem has 10 points**

E.g. MB 19.16 means problem #16 on page 19 in the book by M. Boas, 3rd Edition.

1. MB 147.2 For the given sets of vectors, find the dimension of the space spanned by them and a basis for this space:

   (a) $(1,-1,0,0),\ (0,-2,5,1),\ (1,-3,5,1),\ (2,-4,5,1)$;

   To find the dimension of the space spanned by the set of vectors, we create a matrix where the vectors are the rows components, and we put the matrix in reduced row echelon form:

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & -2 & 5 & 1 \\ 1 & -3 & 5 & 1 \\ 2 & -4 & 5 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -5/2 & -1/2 \\ 0 & 1 & -5/2 & -1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} .$$

   Since there are only two indepedent vectors, the dimension of the vector space is $\boxed{d=2}$. The basis vectors are given by the top two row of the reduced matrix. This is a two dimensional vector space V_2 embedded in a four dimensional vector space V_4 since the vectors have four componets but only two basis vectors. That means the vectors can span a two dimensional plane oriented in four dimensional space.

   (b) $(0,1,2,0,0,4),\ (1,1,3,5,-3,5),\ (1,0,0,5,0,1),\ (-1,1,3,-5,-3,3),\ (0,0,1,0,-3,0)$;

   Following a procedure similar to part a), we find

$$\begin{pmatrix} 0 & 1 & 2 & 0 & 0 & 4 \\ 1 & 1 & 3 & 5 & -3 & 5 \\ 1 & 0 & 0 & 5 & 0 & 1 \\ -1 & 1 & 3 & -5 & -3 & 3 \\ 0 & 0 & 1 & 0 & -3 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 5 & 0 & 1 \\ 0 & 1 & 0 & 0 & 6 & 4 \\ 0 & 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} .$$

   The dimension of the vector space is $\boxed{d=4}$. The basis vectors are given by the first fours row of the reduced row echelon matrix.

   (c) $(0,10,-1,1,10),\ (2,-2,-4,0,-3),\ (4,2,0,4,5),\ (3,2,0,3,4),\ (5,-4,5,6,2)$.

Following a procedure similar to part a), we find

$$\begin{pmatrix} 0 & 10 & -1 & 1 & 10 \\ 2 & -2 & -4 & 0 & -3 \\ 4 & 2 & 0 & 4 & 5 \\ 3 & 2 & 0 & 3 & 4 \\ 5 & -4 & 5 & 6 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} .$$

The dimension of the vector space is $\boxed{d = 4}$. The basis vectors are given by the first fours row of the reduced row echelon matrix.

2. MB 147.4 For each set of basis vectors use the Gram-Schmidt method to find an orthonormal set:

(a) $\mathbf{A} = (0, 2, 0, 0),\ \mathbf{B} = (3, -4, 0, 0),\ \mathbf{C} = (1, 2, 3, 4).$

To find the first vector, we choose one of the vectors and normalize it. In this set, $\mathbf{A}$ is the simplest vector to normalize:

$$\hat{\mathbf{e}}_1 = (0, 1, 0, 0) .$$

Next, we select one of the remaining vectors, in our case we choose $\mathbf{B}$, and we subtract the projection of $\mathbf{B}$ along $\hat{\mathbf{e}}_1$:

$$\mathbf{e}_2 = \mathbf{B} - (\mathbf{B} \cdot \hat{\mathbf{e}}_1)\hat{\mathbf{e}}_1 = (3, 0, 0, 0) .$$

This procedure removes the vector component along $\hat{\mathbf{e}}_1$, thus making $\mathbf{e}_2$ orthogonal to $\hat{\mathbf{e}}_1$. Then we normalize $\mathbf{e}_2$, so that $\hat{\mathbf{e}}_2 = (1, 0, 0, 0)$. Similarly, we can make $\mathbf{C}$ orthogonal to both $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ by subtracting the projection of $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ from $\mathbf{C}$:

$$\mathbf{e}_3 = \mathbf{C} - (\mathbf{C} \cdot \hat{\mathbf{e}}_1)\hat{\mathbf{e}}_1 - (\mathbf{C} \cdot \hat{\mathbf{e}}_2)\hat{\mathbf{e}}_3 = (0, 0, 3, 4) .$$

This normalizes to $\hat{\mathbf{e}}_3 = \frac{1}{5}(0, 0, 3, 4)$ and thus the set of othornomal vectors are

$$\boxed{\hat{\mathbf{e}}_1 = (0, 1, 0, 0),\ \hat{\mathbf{e}}_2 = (1, 0, 0, 0),\ \hat{\mathbf{e}} = \frac{1}{5}(0, 0, 3, 4)} .$$

(b) $\mathbf{A} = (0, 0, 0, 7),\ \mathbf{B} = (2, 0, 0, 5),\ \mathbf{C} = (3, 1, 1, 4)$. Following a similar procedure, we have

$$\boxed{\hat{\mathbf{e}}_1 = (0, 0, 0, 1),\ \hat{\mathbf{e}}_2 = (1, 0, 0, 0),\ \hat{\mathbf{e}} = \frac{1}{\sqrt{2}}(0, 1, 1, 0)} .$$

(c) $\mathbf{A} = (6, 0, 0, 0),\ \mathbf{B} = (1, 0, 2, 0),\ \mathbf{C} = (4, 1, 9, 2)$. Following a similar procedure, we have

$$\boxed{\hat{\mathbf{e}}_1 = (1, 0, 0, 0),\ \hat{\mathbf{e}}_2 = (0, 0, 1, 0),\ \hat{\mathbf{e}} = \frac{1}{\sqrt{5}}(0, 1, 0, 2)} .$$

Keep in mind there are many possible solutions.

3. MB 147.5 Find the norms of **A** and **B** and the inner product of **A** and **B**, and note that the Schwarz inequality is satisfied

   (a)
   $$\mathbf{A} = (3+i, 1, 2-i, -5i, i+1),\ \mathbf{B} = (2i, 4-3i, 1+i, 3i, 1)$$

   The normalization of **A** is $\|\mathbf{A}\| = (\mathbf{A}\cdot\mathbf{A})^{1/2}$. The dot product can thought of as the matrix multiplication of a column vector onto a row vector:

   $$\mathbf{A}\cdot\mathbf{A} = \begin{pmatrix} 3+i & 1 & 2-i & -5i & i+1 \end{pmatrix} \begin{pmatrix} 3+i \\ 1 \\ 2-i \\ -5i \\ i+1 \end{pmatrix} = 43\ .$$

   Note that the row vector is the complex conjugate tranpose of the column vector. Hence, $\boxed{\|\mathbf{A}\| = \sqrt{43}}$.

   The normalization of **B** is

   $$\mathbf{B}\cdot\mathbf{B} = \begin{pmatrix} -2i & 4+3i & 1-i & -3i & 1 \end{pmatrix} \begin{pmatrix} 2i \\ 4-3i \\ 1+i \\ 3i \\ 1 \end{pmatrix} = 46;.$$

   Hence, $\boxed{\|\mathbf{B}\| = \sqrt{46}}$.

   The inner product of $\mathbf{A}\cdot\mathbf{B}$ is

   $$\mathbf{A}\cdot\mathbf{B} = \begin{pmatrix} 3+i & 1 & 2-i & -5i & i+1 \end{pmatrix} \begin{pmatrix} 2i \\ 4-3i \\ 1+i \\ 3i \\ 1 \end{pmatrix} = \boxed{-7+5i}\ .$$

   Thus, $\boxed{|\mathbf{A}\cdot\mathbf{B}| = \sqrt{74}}$.

   The Schwarz inequality is

   $$\boxed{\sqrt{74} = |\mathbf{A}\cdot\mathbf{B}| \le \|\mathbf{A}\|\|\mathbf{B}\| = \sqrt{41}\sqrt{46} = \sqrt{1886}}\ .$$

(b)

$$\mathbf{A} = (2, 2i-3, 1+i, 5i, i-2)$$
$$\mathbf{B} = (5i-2, 1, 3+i, 2i, 4)$$

The normalization of $\mathbf{A}$ is

$$\mathbf{A}\cdot\mathbf{A} = \begin{pmatrix} 2 & -2i-3 & 1-i & -5i & -i-2 \end{pmatrix} \begin{pmatrix} 2 \\ 2i-3 \\ 1+i \\ 5i \\ i-2 \end{pmatrix} = 49\ .$$

Thus, $\boxed{\|\mathbf{A}\| = 7}$. The normalization of $\mathbf{B}$ is

$$\mathbf{B}\cdot\mathbf{B} = \begin{pmatrix} -5i-2 & 1 & 3-i & -2i & 4 \end{pmatrix} \begin{pmatrix} 5i-2 \\ 1 \\ 3+i \\ 2i \\ 4 \end{pmatrix} = 60\ .$$

Thus, $\boxed{\|\mathbf{B}\| = 2\sqrt{15}}$. The dot product of a $\mathbf{A}$ and $\mathbf{B}$ is

$$\mathbf{A}\cdot\mathbf{B} = \begin{pmatrix} 2 & -2i-3 & 1-i & -5i & -i-2 \end{pmatrix} \begin{pmatrix} 5i-2 \\ 1 \\ 3+i \\ 2i \\ 4 \end{pmatrix} = \boxed{2i-1}\ .$$

The Schwarz inequality is

$$\boxed{\sqrt{5} = |\mathbf{A}\cdot\mathbf{B}| \leq \|\mathbf{A}\|\|\mathbf{B}\| = 14\sqrt{15}}\ .$$

4. MB 159.7

(a) Show that if C is a 3×3 matrix whose columns are the components (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) of the three perpendicular vectors each of unit length, then C is an orthogonal matrix.

Recall that for orthogonal matrix, O, that $O^T = O^{-1}$, such that $O^T O = \mathbb{1}$. In matrix form $C^T C$ is

$$\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix} \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix}\ . \tag{1}$$

Notice that each step of matrix multiplication in Eq. (1) is equivalent to taking the dot product of a row and column:

$$x_n x_m + y_n y_m + z_n z_m = \begin{cases} 1 & n = m \\ 0 & n \neq m \ . \end{cases}$$

Hence,

$$C^T C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ ,$$

and C is an orthongal matrix.

(b) Show that if C is an $n \times n$ matrix whose columns are the components $(C_{11}, C_{21}, \ldots, C_{n1})$, $(C_{21}, C_{22}, \ldots, C_{n2})$, $\ldots$ $(C_{1n}, C_{2n}, \ldots, C_{nn})$ of the n perpendicular vectors each of unit length, then C is an orthogonal matrix.

Just as in part a), the matrix C is orthogonal if $C^T C = \mathbb{1}$. In index notation the matrix product is written as

$$(C^T C)_{ij} = \sum_{k=0}^{n} (C^T)_{ik} C_{kj} = \delta_{ij} \ ,$$

where $\sum_{k=0}^{n} (C^T)_{ik} C_{kj}$ is essentially the dot product between unit vectors i and j. Since the unit vectors are all perpendicular to each other, the dot product between any two is zero expect when a vector is dotted with itself. Hence, C is orthogonal.

5. MB 159.18 Find the eigenvalues and eigenvectors of the following matrix:

$$A = \begin{pmatrix} -1 & 1 & 3 \\ 1 & 2 & 0 \\ 3 & 0 & 2 \end{pmatrix} \ ,$$

The eigenvalues are the values of λ that correspond to the zeroes of the characteristic polynomial, i.e. $\det(A - \lambda \mathbb{1}) = 0$. In this case the charactersistic polynomial is $-\lambda^3 + 3\lambda^2 + 10\lambda - 24 = 0$. If we factor this we have $(\lambda - 4)(\lambda - 2)(\lambda + 3) = 0$. Hence, the eigenvalues are

$$\boxed{\lambda = 4, \ 2, \ -3} \ .$$

We find the eigenvectors by solving the set of linear equations given by $A\mathbf{v} = \lambda \mathbf{v}$ for each eigenvalue λ. Recall that there are three possible cases for the solution to a set of equations: no solution, one solution, and infinitely many solutions. For $\lambda = 4$ the set of equations is

$$-5a + b + 3c = 0 \tag{2}$$
$$a - 2b = 0 \tag{3}$$
$$3a - 2c = 0\ . \tag{4}$$

By inspecting Eq. (4), we can see that $c = 3a/2$ and b is free to take on any value and by examining Eq. (3), we can see that $b = a/2$. And a is free to take on any value. Note that $a = 0$ is the trivial solution. Hence, the eigenvector correpsonding to this matrix is

$$\boxed{\mathbf{v}_4 = (1, 1/2, 3/2)a}$$

.

Similarly, for $\lambda = 2$ we have

$$-3a + b + 3c = 0 \tag{5}$$
$$a = 0 \tag{6}$$
$$3a = 0\ . \tag{7}$$

By inspecting Eq. (6) and Eq. (7), we can see that $a = 0$, b and c are free to take on any value and by examining Eq. (5), we can see that $c = -b/3$. Hence, the eigenvector correpsonding to this matrix is

$$\boxed{\mathbf{v}_2 = (0, 1, -1/3)b}\ .$$

For $\lambda = -3$

$$2a + b + 3c = 0 \tag{8}$$
$$a - b = 0 \tag{9}$$
$$3a - c = 0\ . \tag{10}$$

By inspecting Eq. (10), we can see that $c = 3a$ and b is free to take on any value and by examining Eq. (9), we can see that $b = a$. Hence, the eigenvector corresponding to this matrix is

$$\boxed{\mathbf{v}_{-3} = (1, 1/2, 3/2)a}\ .$$

6. MB 159.19 Find the eigenvalues and eigenvectors of the following matrix:

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 3 & 0 \\ 2 & 0 & 3 \end{pmatrix}\ ,$$

Following a similar procedure to that described for problem MB 159.18, we find the eigenvalues to be $\boxed{\lambda = 5, 3, -1}$ and the normalized eigenvectors are respectively:

$$\mathbf{v}_5 = \frac{1}{\sqrt{3}}(1,1,1)$$
$$\mathbf{v}_3 = \frac{1}{\sqrt{2}}(0,-1,1)$$
$$\mathbf{v}_{-1} = \frac{1}{\sqrt{6}}(-2,1,1) \ .$$

7. MB 160.42

Verify that the matrix is Hermitian. Find its eigenvalues and eigenvectors, write a unitary matrix U which diagonalizes H by a similarity transformation, and show that $U^{-1}HU$ is the diagonal matrix of eigenvalues.

$$H = \begin{pmatrix} 3 & 1-i \\ 1+i & 2 \end{pmatrix} \ .$$

To show that the matrix is Hermitian we take the conjugate transpose of the matrix and show that $H = H^\dagger$:

$$\boxed{H^\dagger = \begin{pmatrix} 3 & 1-i \\ 1+i & 2 \end{pmatrix}} \ .$$

To find the eigenvalues, we compute the zeroes of the characteristic polynomial, $\det(H - \lambda\mathbf{1}) = 0$:

$$\det(H - \lambda\mathbf{1}) = \begin{vmatrix} 3-\lambda & 1-i \\ 1+i & 2-\lambda \end{vmatrix} = \lambda^2 - 5\lambda + 4 = (\lambda-4)(\lambda-1) = 0 \ ,$$

so the eigenvalues are $\lambda = 4, 1$. The eigenvectors are the characteristic vectors that satitfy the relation $(H - \lambda\mathbf{1}))\mathbf{v} = \mathbf{0}$ for $\lambda = 4, 1$ where $\mathbf{v} = (a, b)$. Usually, we can solve this by inspecion; however, we can find the eigenvector systematic manner, by writting $(H - \lambda\mathbf{1}))\mathbf{v} = \mathbf{0}$ as an augmented matrix and then tranforming the matrix into reduced row echelon form:

$$\begin{pmatrix} -1 & 1-i & 0 \\ 1+i & -2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -(1-i) & 0 \\ 0 & 0 & 0 \end{pmatrix} \ .$$

(Keep in mind that on an exam it is usually faster to find the eigenvector of a 2×2 or 3×3 matrix by inspection, i.e. guess and check as we showed in MB 159.18.)

Now, we have the equation $-a + (1+i)b = 0$, which tells that an eigenvector is $\mathbf{v}_4 = (1, (1+i)/2)a$ and the normalized eigenvector is

$$\boxed{\hat{\mathbf{v}}_4 = \frac{1}{\sqrt{3/2}}(1, (1+i)/2)} \ .$$

Using a similar procedure for $\lambda = 1$, we have

$$\begin{pmatrix} 2 & 1-i & 0 \\ 1+i & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & (1-i)/2 & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

This tells that $a + (1-i)b/2 = 0$ an eigenvector is $\mathbf{v}_1 = (1, -(1+i))a$ and the normalized eigenvector is

$$\boxed{\hat{\mathbf{v}}_1 = \frac{1}{\sqrt{3}}(1, -(1+i))} .$$

The unitary matrix is constructed from the normalized eigenvectors:

$$\boxed{U = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3/2} \\ -(1+i/\sqrt{3}) & -(1+i)/\sqrt{3/2} \end{pmatrix}} ,$$

and the inverse is just the conjugate transpose (if and only if the matrix is Hermitian)

$$U^\dagger = \begin{pmatrix} 1/\sqrt{3} & -(1-i)/\sqrt{3} \\ 1/\sqrt{3/2} & (1-i)/2\sqrt{3/2} \end{pmatrix} .$$

Now we can show that

$$\boxed{U^\dagger H U = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}} .$$

8. Find the right and left eigenvectors the following matrix:

$$M = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix} .$$

We find eigenvalues to be$\lambda = 4, -1$ by solving for the zeroes of the characteristic polynomial: $\det(M - \lambda(1)) = 0$.

i) The right eigenvectors satisfy the relation: $M\mathbf{v} = \lambda\mathbf{v}$ where $\mathbf{v} = (a, b)$ is column vector. For $\lambda = 4$, the augmented matrix is

$$\begin{pmatrix} -3 & 3 & 0 \\ 2 & -2 & 0 \end{pmatrix}$$

Row reduce the matrix to get

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

This gives us the equation $-3a + 3b = 0$, which tells us that the first normalized right eigenvector

$$\boxed{\hat{\mathbf{v}}_{R,4} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}}\,.$$

Similarly, for $\lambda = -1$, the augemented matrix is

$$\begin{pmatrix}2 & 3 & 0\\2 & 3 & 0\end{pmatrix}\,.$$

Row reduce the matrix to get:

$$\begin{pmatrix}1 & 3/2 & 0\\0 & 0 & 0\end{pmatrix}\,.$$

From the equation $a + 3/2b = 0$ we get the second normalized right eigenvector

$$\boxed{\hat{\mathbf{v}}_{R,-1} = \frac{1}{\sqrt{11/2}}\begin{pmatrix}1\\2/3\end{pmatrix}}\,.$$

ii) The left eigenvectors satisfy the relation: $\mathbf{v}M = \mathbf{v}\lambda$ where $\mathbf{v} = (a, b)$ is row vector. For $\lambda = 4$, the matrix relation is

$$\begin{pmatrix}a & b\end{pmatrix}\begin{pmatrix}-3 & 3\\2 & -2\end{pmatrix} = \begin{pmatrix}4a & 4b\end{pmatrix}\,.$$

Of course, we can solve this by inspection, but for more difficult cases a systematic method is useful. If we take the tranpose of both side of the matrix relation, we can put this in a more familiar form:

$$\begin{pmatrix}1 & 2\\3 & 2\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}4a\\4b\end{pmatrix}$$

Now, we can write this as an augmented matrix:

$$\begin{pmatrix}-3 & 2 & 0\\3 & -2 & 0\end{pmatrix}\,,$$

and we can row reduce the matrix to get

$$\begin{pmatrix}1 & -2/3 & 0\\0 & 0 & 0\end{pmatrix}\,.$$

This is gives us the relation $a - (2/3)b = 0$, so we can see normalized left vector is

$$\boxed{\hat{\mathbf{v}}_{L,4} = \frac{1}{\sqrt{11/2}}(1, 3/2)}\,.$$

Using the procedure described above, we find the left eigenvector for $\lambda = -1$ to be

$$\boxed{\hat{\mathbf{v}}_{L,-1} = \frac{1}{\sqrt{2}}(1, 1)}\,.$$

9. Consider the matrix

$$M = \begin{bmatrix} 2 & 1+\epsilon \\ 1-\epsilon & 3 \end{bmatrix}$$

Calculate its right and left eigenvectors for an arbitrary ϵ. Show that these coincide when $\epsilon \to 0$. Find the value of ϵ below which the eigenvalues become real.

First we find the eigenvalues by solving the zeroes of the characteristic polynomial, $\det(M - \lambda\mathbf{1}) = 0$:

$$\begin{vmatrix} 2-\lambda & 1+\epsilon \\ 1-\epsilon & 3-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda) - (1-\epsilon)(1+\epsilon) = 0$$

Next, if we write this relation in powers of λ, we get

$$\lambda^2 - 5\lambda + 6 - (1-\epsilon^2) = 0 \; .$$

Now, we can write this as a quadratic equation

$$\lambda_\pm = \frac{5}{2} \pm \frac{\sqrt{5-4\epsilon^2}}{2} \; .$$

We can see that for $\boxed{|\epsilon| < \sqrt{5/4}}$ the eigenvalues are real.

i) The right eigenvectors satisfy the relation $M\mathbf{v} = \lambda\mathbf{v}$. For λ_+, by inspecting the equations:

$$\begin{aligned} (-1/2 - \sqrt{5-4\epsilon^2}/2)a + (1+\epsilon)b = 0 \\ (1-\epsilon)a + (1/2 - \sqrt{5-4\epsilon}/2)b = 0 \; , \end{aligned}$$

we can see that $\mathbf{v}_{R,+} = (-1/2 + \sqrt{5-4\epsilon^2}/2, 1-\epsilon)$ is the right eigenvector. For λ_-, by inspecting the equations:

$$\begin{aligned} (-1/2 + \sqrt{5-4\epsilon^2}/2)a + (1+\epsilon)b = 0 \\ (1-\epsilon)a + (1/2 + \sqrt{5-4\epsilon}/2)b = 0 \; , \end{aligned}$$

we can see that $\mathbf{v}_{R,-} = (-1/2 - \sqrt{5-4\epsilon^2}/2, 1-\epsilon)$ is the other eigenvector.

ii) The left eigenvectors satisfy the relation $\mathbf{v}M = \mathbf{v}\lambda$. For λ_+, we have the equations:

$$\begin{aligned} (-1/2 - \sqrt{5-4\epsilon^2}/2)a + (1-\epsilon)b = 0 \\ (1+\epsilon)a + (1/2 - \sqrt{5-4\epsilon}/2)b = 0 \; . \end{aligned}$$

and by inspection we can see that $\mathbf{v}_{L,+} = (-1/2 - \sqrt{5-4\epsilon^2}/2, 1+\epsilon)$ is a left eigenvector. For λ_-, we have the equations:

$$
\begin{aligned}
(-1/2 + \sqrt{5-4\epsilon^2}/2)a + (1-\epsilon)b = 0 \\
(1+\epsilon)a + (1/2 + \sqrt{5-4\epsilon}/2)b = 0 \ .
\end{aligned}
$$

and by inspection we can see that $\mathbf{v}_{L,-} = (-1/2 - \sqrt{5-4\epsilon^2}/2, 1+\epsilon)$ is the eigenvector left eigenvector. As we can see as $\epsilon \to 0$, the right and left eigenvalue are the same:

$$
\boxed{
\begin{aligned}
\mathbf{v}_{R,+} = \mathbf{v}_{L,+} = (-1/2 - \sqrt{5}/2, 1) \\
\mathbf{v}_{R,+} = \mathbf{v}_{L,-} = (-1/2 + \sqrt{5}/2, 1) \ .
\end{aligned}
}
$$