Mathematical Methods of Physics 116A- Winter 2018

Physics 116A

Home Work # 9 Solutions
Posted on Mar 8, 2018
Due in Class Mar 15, 2018

§**Required Problems: Each problem has 10 points**

E.g. MB 19.16 means problem #16 on page 19 in the book by M. Boas, 3rd Edition.

1. MB 171.2 For the conic

$$2x^2 + 4xy - y^2 = 24$$

find the equation relative to the principal axes.

In matrix form this can be written as

$$x^2 + 2xy + 2yx - y^2 = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 24$$

We can find the principal axes by diagonalizing the matrix,

$$M = \begin{pmatrix} 2 & -2 \\ -2 & 1 \end{pmatrix} ,$$

using a similarity transformation, i.e. $C^{-1}MC = D$. Following the methods described in MB section 11, we find eigenvalues to be $\lambda = 3, -2$ and the corresponding normalized eigenvectors are

$$\hat{\mathbf{v}}_{\lambda=3} = \frac{1}{\sqrt{5}}(1, -2)$$
$$\hat{\mathbf{v}}_{\lambda=-2} = \frac{1}{\sqrt{5}}(2, -1) .$$

From the normalized eigenvector we can construct an invertable matrix

$$C = 1/\sqrt{5} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} ,$$

such that

$$C^{-1}MC = D = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix} .$$

Hence, the conic the equation along the principal axes is

$$\boxed{\begin{pmatrix} x' & y' \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = 3x'^2 - 2y'^2 = 24} .$$

2. MB 171.5 For the quadratic surface

$$F(x, y, z) = 5x^2 + 3y^2 + 2z^2 + 4xz = 14$$

find the equation relative to the principal axes.

This equation can be expanded as

$$\begin{aligned} F(x, y, z) &= c_{11}x^2 + c_{12}xy + c_{13}xz \\ &\quad + c_{21}yz + c_{22}y^2 + c_{23}yz \\ &\quad + c_{31}zx + c_{32}zy + c_{33}z^2 = b \ , \end{aligned} \tag{1}$$

where $c_{ij} = c_{ji}$ and b is a constant. In matrix form this can be written as

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = b \ .$$

Thus we have

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 5 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 14 \ . \tag{2}$$

The eigenvalues of the matrix in Eq. (2) are $\lambda = 6, 3, 1$ and the coorespond-ing normalized eigenvectors are the columns of following transformation matrix

$$C = \begin{pmatrix} 2/\sqrt{5} & 0 & -1/\sqrt{5} \\ 0 & 1 & 0 \\ 1/\sqrt{5} & 0 & 2/\sqrt{5} \end{pmatrix}$$

The similarity transformation is

$$\begin{aligned} C^{-1}MC &= D \\ &= \begin{pmatrix} x' & y' & z' \end{pmatrix} \begin{pmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \\ &= \boxed{6x'^2 + 3y'^2 + z'^2 = 14} \ . \end{aligned}$$

Note that any interchanging of the labels x', y', and z' is also correct.

3. MB 172.12 Verify the details of Example 5, equations (12.16) to (12.36): In Example 5. Let's consider a model of linear triatomic molecule in which we approximate the forces between the atoms by forces due to springs (MB Figure 12.2) given that $(m'_1, k'_1, m'_2, k'_2, m'_3) = (m, k, M, k, m)$

   The potential energy of the system is given by

$$V = \frac{1}{2}k'_1(x - y)^2 + \frac{1}{2}k'_2(y - z)^2 \ .$$

The equation of the motion is given by

$$\begin{cases} m_1'\ddot{x} = -\partial V/\partial x \\ m_2'\ddot{y} = -\partial V/\partial y \\ m_3'\ddot{z} = -\partial V/\partial z \end{cases}$$

This is a set of linear differential equations with solutions $x = x_0 e^{i\omega t}$, $y = y_0 e^{i\omega t}$, and $z = z_0 e^{i\omega t}$. Differentiating this we find the following set linear equations:

$$\begin{cases} -m_1'\omega^2 x = -k_1'(x-y) \\ -m_2'\omega^2 y = k_1'(x-y) - k_2'(y-z) \\ -m_3'\omega^2 z = k_2'(y-z) \end{cases} . \tag{3}$$

Now, we can plug the given values into Eq. (3) to find

$$\begin{cases} -m\omega^2 x = -k(x-y) \\ -M\omega^2 y = k(x-y) - k(y-z) \\ -m\omega^2 z = k(y-z) \end{cases} , \tag{4}$$

and we can write the linear system in matrix form as

$$\frac{m\omega^2}{k}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ -m/M & 2m/M & -m/M \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} . \tag{5}$$

Recall that $A\mathbf{r} = \lambda\mathbf{r}$ is an eigenequation, where in this case $\lambda = m\omega^2/k$ and $\mathbf{r} = (x, y, z)$. First find the eigenvalues by solving the charateristic equations $\det(A - \lambda\mathbb{1}) = 0$ which yeilds $\lambda = 0, 1, 1 + 2m/M$. Now, we can solve for the characteristic frequencies ω from the eigenvalues:

$$\boxed{\begin{aligned} &\omega_1 = 0 \\ &\omega_2 = \sqrt{\frac{k}{m}} \\ &\omega_3 = \sqrt{\frac{(1+2m/M)k}{m}} \end{aligned}} .$$

The characteristic modes are related to the eigenvectors:

$$\boxed{\begin{aligned} \mathbf{r}_{\omega_1} &= \frac{1}{\sqrt{3}}(1,1,1) \\ \mathbf{r}_{\omega_2} &= \frac{1}{\sqrt{2}}(1,0,1) \\ \mathbf{r}_{\omega_2} &= \frac{1}{\sqrt{2+(2m/M)^2}}(1,-2m/M,1) \end{aligned}} .$$

4. MB 172.14 Find the characteristic frequencies and the charateristic modes of vibration for a system with (MB Fig. 12.1) given that $(k_1', m_1', k_2', m_2', k_3') = (k, m, 2k, m, k)$.

The potential energy of the system is given by

$$V = \frac{1}{2}k_1' + \frac{1}{2}k_2'(x-y)^2 + \frac{1}{2}k_3' y^2 \ .$$

The equation of the motion is given by

$$\begin{cases} m_1'\ddot{x} = -\partial V/\partial x \\ m_2'\ddot{y} = -\partial V/\partial y \ . \end{cases}$$

This is a set of linear differential equations with solutions $x = x_0 e^{i\omega t}$ and $y = y_0 e^{i\omega t}$. Differentiating this we find the following set linear equations:

$$\begin{cases} -m_1'\omega^2 x = -k_1' x - k_2'(x-y) \\ -m_2'\omega^2 y = -k_3' y + k_2'(x-y) \end{cases} \ . \tag{6}$$

Now, we can plug the given values into Eq. (6) to find

$$\begin{cases} -m\omega^2 x = -kx - 2k(x-y) \\ -m\omega^2 y = -ky + 2k(x-y) \end{cases} \ , \tag{7}$$

and we can write the linear system in matrix form as

$$\frac{m\omega^2}{k}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \ . \tag{8}$$

Recall that $M\mathbf{r} = \lambda\mathbf{r}$ is an eigenequation, where in this case $\lambda = m\omega^2/k$ and $\mathbf{r} = (x, y)$. First find the eigenvalues by solve the charateristic equations $\det(M - \lambda\mathbb{1}) = 0$ which yeilds $\lambda = 5, 1$. Now, we can find the characteristic frequencies ω by solving the eigenvalues

$$\boxed{\begin{aligned} \omega_1 &= \sqrt{\frac{5k}{m}} \\ \omega_2 &= \sqrt{\frac{k}{m}} \end{aligned}} \ .$$

The characteristic modes are related to the eigenvectors:

$$\boxed{\begin{aligned} \mathbf{r}_{\omega_1} &= \frac{1}{\sqrt{2}}(-1, 1) \\ \mathbf{r}_{\omega_2} &= \frac{1}{\sqrt{2}}(1, 1) \end{aligned}} \ .$$

In a characteristic mode (normal mode) each object oscillates at the same frequency and the eigenvectors tells us the porpotionality between the amplitudes of oscillation for each object. In the case of mass-spring system the objects are masses and the amplitudes are the displacements from the equilibrium position.

5. MB 172.16

   Find the characteristic frequencies and the charateristic modes of vibration for a system with (MB Fig. 12.1) given that $(k'_1, m'_1, k'_2, m'_2, k'_3) = (4k, m, 2k, m, k)$.

   The potential energy of the system is given by

$$V = \frac{1}{2}k'_1 + \frac{1}{2}k'_2(x-y)^2 + \frac{1}{2}k'_3 y^2 \ .$$

   The equation of the motion is given by

$$\begin{cases} m'_1\ddot{x} = -\partial V/\partial x \\ m'_2\ddot{y} = -\partial V/\partial y \ . \end{cases}$$

   This is a set of linear differential equations with solutions $x = x_0 e^{i\omega t}$ and $y = y_0 e^{i\omega t}$. Differentiating this we find the following set linear equations:

$$\begin{cases} -m'_1\omega^2 x = -k'_1 x - k'_2(x-y) \\ -m'_2\omega^2 y = -k'_3 y + k'_2(x-y) \end{cases} \ . \tag{9}$$

   Now, we can plug the given values into Eq. (9) to find

$$\begin{cases} -m\omega^2 x = -4kx - 2k(x-y) \\ -m\omega^2 y = -ky + 2k(x-y) \end{cases} \ , \tag{10}$$

   and we can write the linear system in matrix form as

$$\frac{m\omega^2}{k}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 & -2 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \ . \tag{11}$$

   Recall that $M\mathbf{r} = \lambda\mathbf{r}$ is an eigenequation, where in this case $\lambda = m\omega^2/k$ and $\mathbf{r} = (x, y)$. First find the eigenvalues by solving the charateristic equations $\det(M - \lambda\mathbb{1}) = 0$ which yeilds $\lambda = 7, 2$. Now, we can find the characteristic frequencies ω from the eigenvalues

$$\boxed{\begin{array}{l} \omega_1 = \sqrt{\dfrac{7k}{m}} \\ \omega_2 = \sqrt{\dfrac{2k}{m}} \end{array}} \ .$$

The characteristic modes are related to the eigenvectors:

$$\boxed{\begin{aligned}\mathbf{r}_{\omega_1} &= \frac{1}{\sqrt{5}}(-2,1)\\ \mathbf{r}_{\omega_2} &= \frac{1}{\sqrt{5}}(1,2)\end{aligned}},$$

6. MB 172.20 Find the characteristic frequencies and the characteristic modes of vibration as in MB 171. Example 7 for them system in diagram MB Fig. 12.3 given $(k_1', m_1', k_2', m_2') = (3k, 4m, k, m)$

The kinetic energy is

$$T = \frac{1}{2}(m_1'\dot{x} + m_2'\dot{y})$$

The potential energy is

$$V = \frac{1}{2}[k_1'x^2 + k_2(x-y)^2] \ .$$

The equation of the motion is given by

$$\begin{cases} m_1'\ddot{x} = -\partial V/\partial x \\ m_2'\ddot{y} = -\partial V/\partial y \ . \end{cases}$$

This is a set of linear differential equations with solutions $x = x_0 e^{i\omega t}$ and $y = y_0 e^{i\omega t}$. Differentiating this we find the following set linear equations:

$$\begin{cases} -m_1'\omega^2 x = -k_1'x - k_2'(x-y) \\ -m_2'\omega^2 y = k_2'(x-y) \end{cases} . \tag{12}$$

Now, we can plug the given values into Eq. (12) to find

$$\begin{cases} -4m\omega^2 x = -3kx - k(x-y) \\ -m\omega^2 y = k(x-y) \end{cases} ,$$

and we can write the linear system in matrix form as

$$\frac{m\omega^2}{k}\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} .$$

where in this case $\lambda = m\omega^2/k$. In compact form this is $\lambda T\mathbf{r} = V\mathbf{r}$. We can make this into an eigenequation, by multipling from the left both sides of the equation by of both sides by T^{-1} sucht that we have $\lambda\mathbf{r} = T^{-1}V\mathbf{r}$. and $\mathbf{r} = (x, y)$. Note that

$$T^{-1}V = \begin{pmatrix} 1 & -1/4 \\ -1 & 1 \end{pmatrix} .$$

Now, we can find the characteristic frequencies ω from the eigenvalues

$$\boxed{\begin{aligned}\omega_1 &= \sqrt{\frac{3k}{2m}} \\ \omega_2 &= \sqrt{\frac{1k}{2m}}\end{aligned}}\,.$$

The characteristic modes are related to the eigenvectors:

$$\boxed{\begin{aligned}\mathbf{r}_{\omega_1} &= \frac{1}{\sqrt{5}}(-1,2) \\ \mathbf{r}_{\omega_2} &= \frac{1}{\sqrt{5}}(2,1)\end{aligned}}\,.$$

7. MB 172.21 Find the characteristic frequencies and the characteristic modes of vibration as in MB 171. Example 7 for them system in diagram MB Fig. 12.3 given $(k_1', m_1', k_2', m_2') = (3k, m, 2k, m)$

   The kinetic energy is

   $$T = \frac{1}{2}(m_1'\dot{x} + m_2'\dot{y})$$

   The potential energy is

   $$V = \frac{1}{2}[k_1'x^2 + k_2'(x-y)^2]\,.$$

   The equation of the motion is given by

   $$\begin{cases} m_1'\ddot{x} = -\partial V/\partial x \\ m_2'\ddot{y} = -\partial V/\partial y\,. \end{cases}$$

   This is a set of linear differential equations with solutions $x = x_0 e^{i\omega t}$ and $y = y_0 e^{i\omega t}$. Differentiating this we find the following set linear equations:

   $$\begin{cases} -m_1'\omega^2 x = -k_1'x - k_2'(x-y) \\ -m_2'\omega^2 y = k_2'(x-y) \end{cases}\,. \tag{13}$$

   Now we can find the eigenvalues by solving the charateristic equations $\det(T^{-1}V - \lambda\mathbb{1}) = 0$ which yeilds $\lambda = 6, 1$. Now, we can plug the given values into Eq. (13) to find

   $$\begin{cases} -m\omega^2 x = -3kx - 2k(x-y) \\ -m\omega^2 y = 2k(x-y) \end{cases}\,, \tag{14}$$

   and we can write the linear system in matrix form as

   $$\frac{m\omega^2}{k}\begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix}x \\ y\end{pmatrix} = \begin{pmatrix}5 & -2 \\ -2 & 2\end{pmatrix}\begin{pmatrix}x \\ y\end{pmatrix}\,. \tag{15}$$

where in this case $\lambda = m\omega^2/k$. In compact form this is $\lambda T\mathbf{r} = V\mathbf{r}$. We can make this into an eigenequation, by multipling from the left both sides of the equation by of both sides by T^{-1} sucht that we have $\lambda\mathbf{r} = T^{-1}V\mathbf{r}$. and $\mathbf{r} = (x, y)$.

Now we can find the eigenvalues by solving the charateristic equations $\det\left(T^{-1}V - \lambda\mathbb{1}\right) = 0$ which yeilds $\lambda = 6, 1$. Now, we can find the characteristic frequencies ω from the eigenvalues

$$\boxed{\begin{aligned}\omega_1 &= \sqrt{\frac{6k}{m}}\\ \omega_2 &= \sqrt{\frac{k}{m}}\end{aligned}}\,.$$

The characteristic modes are related to the eigenvectors:

$$\boxed{\begin{aligned}\mathbf{r}_{\omega_1} &= \frac{1}{\sqrt{5}}(-2, 1)\\ \mathbf{r}_{\omega_2} &= \frac{1}{\sqrt{5}}(1, 2)\end{aligned}}\,.$$

8. MB 177.4 Show that the matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},$$

under matrix multiplication, form a group. Write the group multiplication table to see that this group (called the 4's group) is not isomorphic to the cyclic group of order 4 in Problem 1. Show that the 4's group is Abelian but not cyclic.

The multiplication table for the 4's group is

	I	A	B	C
I	I	A	B	C
A	A	I	C	B
B	B	C	I	A
C	C	B	A	I

.

Now let's check the set equipped with matrix multiplication forms a group:

(i) Closure: every row and column spans the set, so it is closed. ✓

(ii) Associativity: Matrix multiplication is associative. ✓

(iii) Idenity: there is an idenity matrix in the set. ✓

(iv) Inverse: each element is its own inverse. ✓

Since the four axioms are satisfied, the set equipped with matrix multipication is a group

If we label $A = i$, $B = A^2 = -1$, $C = A^3 = -i$, $I = A^4 = 1$, we can compare the multilication table for the cyclic group with that of the 4's group:

	I	A	B	C
I	I	A	B	C
A	A	B	C	I
B	B	C	I	A
C	C	I	A	B

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	I	A	B	C
I	I	A	B	C
A	A	I	C	B
B	B	C	I	A
C	C	B	A	I

and we can see that tables are different; Hence, the groups are not isomorphic.

To show that the 4's group is *not* cyclic, we show that there does not exist an $A \in X$, where X is the set of all elements in the 4's group, such that $\{A, A^2, A^3, A^4\}$ is the complete set. Since $\forall\ A \in X,\ A^2 = I$, the set is not cycic.

9. MB 178.5

Consider the group 4 with unit element I and other elements A, B, C, where $AB = BA = C$, and $A^2 = B^2 = I$. Write the group multiplication table and verify that it is a group. There are two groups of order 4: To which is this one isomorphic?

To construct the matrix table we need to know results of AC, CA, BC, CB, and CC. Let's compute $AC = AAB = IB = B$, $CA = BAA = BI = B$, $BC = BBA = IA = A$, $CB = ABB = AI = A$ and $CC = ABBA = AIA = AA = I$.

	I	A	B	C
I	I	A	B	C
A	A	I	C	B
B	B	C	I	A
C	C	B	A	I

Now let's check the set equipped with matrix multiplication forms a group:

(i) Closure: every row and column spans the set, so it is closed. ✓

(ii) Associativity: Matrix multiplication is associative. ✓

(iii) Idenity: there is an idenity matrix in the set. ✓

(iv) Inverse: each element is its own inverse. ✓

Since the four axioms are satisfied, the set equipped with matrix multipication is a group.

We can see that this is isomorphic to the 4's group.