Mathematical Methods of Physics

Physics 116A- Winter 2018

Midterm Examination Solutions, **Total 100 Points**
Feb 13, 2018

1. Consider the series

$$S_A = \sum_{n=2}^{\infty} (-1)^n \frac{1}{n(\log n)^s}$$

and the corresponding absolute series

$$S_B = \sum_{n=2}^{\infty} \frac{1}{n(\log n)^s}$$

a) Determine if S_A converges with s=1.

Check using the alternating series test:

$$\left|\frac{1}{(n+1)\ln(n+1)}\right| \leq \left|\frac{1}{n(\ln n)}\right| \quad \forall\, n \geq 2$$

and

$$\lim_{n\to\infty} \left|\frac{1}{n(\ln n)}\right| = 0\ .$$

$$\boxed{\text{Since both conditions are true, the series converges.}}$$

b) What is a bound on the error if we take the partial sum S_A, by keeping only the first 4 terms? ...[10]

An error bound is defined as an upper bound for the expression $|S - S_N|$ where S is the limit of the series and S_N is the partial sum. For an alternating series a convenient choice for the error bound when $n \geq N$ is given by $|a_{N+1}|$. In our case the fourth term is $N = 5$, so the error bound is

$$\epsilon = \left|\frac{1}{6(\ln 6)^s}\right|\ .$$

c) Find the range of s where the series S_B converges. ...[15]

According to the integral test, if

$$\int^{\infty} \frac{1}{n(\ln n)^2} \mathrm{d}n$$

converges(diverges), then S_B converges(diverges). We can integrate easily if we make the substitution $y = \ln n$ and $\mathrm{d}y = (1/n)\mathrm{d}n$:

$$\int^{\infty} y^{-s}\mathrm{d}y = \left.\frac{y^{-s+1}}{-s+1}\right|^{\infty}.$$

We can immediately see that the series diverges at $s = 1$ because the denominator is zero and the series diverges for $s < 1$ because the numerator blows up as $y \to \infty$, but everywhere else it is convergent. $\boxed{\text{Hence, the series is convergent for } s > 1}$.

{ Hint: Here we take the natural logarithm. For the Part (c) the integral test is well suited. }

2. In Fig. (1) we have a circuit connected to an AC power source producing a potential difference of V between points A and B. We can think of current in a circuit like water flow in a pipe, where the voltage difference in a circuit corresponds to the pressure difference in a pipe. For example, when the water flow meets a junction in a pipe, the water flow will divide amoung the branches, but the total amount of water flow will remain constant.

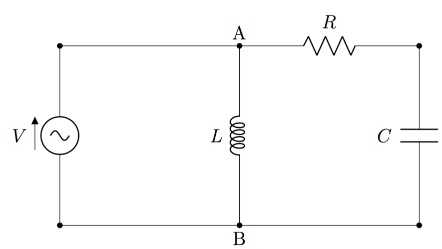

Figure 1:

a) Use the generalization of Ohm's law to find the ratio (I_{RC}/I_L) of current going through the RC branch with respect to the L branch. . . . [10]

According to Kirchoff's voltage law, the voltage difference between any two branches that connect to the same two point are equal, i.e. $V_{RC} = V_L$. If we use Kirchoff's voltage together Ohm's law ($V = IZ$), we can write that $I_{RC}Z_{RC} = I_L Z_L$.

$$\boxed{\text{Hence, } \frac{I_{RC}}{I_L} = \frac{Z_L}{Z_{RC}} = \frac{Z_L}{Z_R + Z_C}}.$$

b) Use the result of part a) to find the impedence in terms of Z_R, Z_L and Z_C. . . . [10]

From Kirchoff's voltage law, we know that the voltage difference across the power source is equal the voltage difference accross the inductor ($V = V_L$). Using Kirchoff's voltage law together with Ohm's law, we can write an equation for the effective impedence:

$$Z = \frac{I_L Z_L}{I} \ . \tag{1}$$

Kirchoff's current law states that the current entering a junction is exactly equal to current exiting a junction, i.e. $I = I_{RC} + I_L$. Using Kirchoff's current law together with the result of part a) we can find an expression for the total current in terms I_L is

$$I = I_L \left(1 + \frac{Z_L}{Z_R + Z_C}\right) \ .$$

Finally, we plug this result into Eq. (1) and simplify to find

$$\boxed{Z = \frac{Z_L(Z_R + Z_C)}{Z_R + Z_L + Z_C}} \ . \tag{2}$$

c) Express ω in terms of R, L, and C if the angle of Z is 45°. ... [10]

The first step is plug $Z_R = R$, $Z_L = i\omega L$, and $Z_C = 1/(i\omega C)$ into Eq. (2) and to convert Z into rectangular form:

$$\begin{aligned} Z &= \frac{i\omega L(R + 1/(i\omega C))}{R + i(\omega L - 1/(\omega C))} \\ &= \frac{(i\omega LR + L/C)}{R + i(\omega L - 1/(\omega C))} \frac{(R - i(\omega L - 1/(\omega C)))}{(R - i(\omega L - 1/(\omega C)))} \\ &= \frac{(i\omega LR + L/C)(R - i(\omega L - 1/(\omega C))}{R^2 + (\omega L - 1/(\omega C))^2} \\ &= \frac{\omega^2 L^2 R + i(L/\omega C^2 - \omega L^2/C + \omega LR^2)}{R^2 + (\omega L - 1/(\omega C))^2} \ . \end{aligned} \tag{3}$$

Recall that $\tan\theta = \operatorname{Im}\{Z\}/\operatorname{Re}\{Z\}$, so for $\theta = 45°$ we have

$$\tan 45° = \frac{(L/\omega C^2 - \omega L^2/C + \omega LR^2)}{\omega^2 L^2 R} = 1 \ .$$

First we multiply the numerator and the denominator by ωC^2, and next we multiply both sides of the equation by $\omega^2 L^2 R$, in order to simplify the expression by eliminating all the factions:

$$(L - \omega^2 L^2 C^2 + \omega^2 LR^2 C^2) = \omega^3 L^2 RC^2 \ . \tag{4}$$

Finally, we substract the LHS from the RHS of Eq. (4) and simplify by expanding the equation in powers of ω:

$$\boxed{\omega^3 L^2 RC^2 + \omega^2(L^2 C - LC^2 R^2) - L = 0} \ .$$

d) Find the resonance frequency ω. ... [5]

Resonance occurs when the imaginary part of the impedence vanishes, i.e. $\mathrm{Im}\{Z\} = 0$. In our case

$$\mathrm{Im}\{Z\} = (L/\omega C^2 - \omega L^2/C + \omega L R^2) = 0 \ .$$

We begin by multiplying both sides by $-\omega C^2$, which removes all the fractions:

$$(-L + \omega^2 L^2 C - \omega^2 L R^2 C^2) = 0 \ .$$

Next we simplify the expression by expanding it in powers of ω:

$$\omega^2 (L^2 C - L R^2 C^2) - L = 0 \ .$$

Finally, we notice that this is a quadratic equation which we can solve for explicitly:

$$\boxed{\omega = \pm\sqrt{\frac{L}{L^2C^2 - LR^2C^2}} = \pm\sqrt{\frac{1}{LC - R^2C^2}}} \ .$$

3. Express this set of equations as an augmented matrix and use row reduction to determine if it is consistent:

$$\begin{cases} x + 2y + 3z = 9 \\ 5x - 11y + 5z = 45 \\ x - 9z = 5 \end{cases} \ .$$

If it is consistent find the solution or solutions to the equations, and find the rank of the *augmented* matrix. ... [30]

The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 5 & -11 & 5 & 45 \\ 1 & 0 & -9 & 5 \end{array}\right) \ .$$

We begin by row reducing all the elements below the pivot element (marked by a box) of R_1:

$$\left(\begin{array}{ccc|c} \boxed{1} & 2 & 3 & 9 \\ 5 & -11 & 5 & 45 \\ 1 & 0 & -9 & 5 \end{array}\right) \xrightarrow[R_2 \to R_2 - 5R_1]{R_3 \to R_3 - R_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & -21 & -10 & 0 \\ 0 & -2 & -12 & -4 \end{array}\right) \ .$$

Next we row reduce all the element below the pivot element of the R_2:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & \boxed{-21} & -10 & 0 \\ 0 & -2 & -12 & -4 \end{array}\right) \xrightarrow{R_3 \to R_3 - (2/21) R_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & -21 & -10 & 0 \\ 0 & 0 & -232/21 & -4 \end{array}\right) \ .$$

Let's pause here for moment. Notice that if we put the augmented matrix back into equation form, we can very easily solve z, then we can plug z into the second equation and solve for y, and then we can plug both z and y into the first equation and solve for x. Now if we wanted put the matrix into reduced row echelon form, we can continue by setting the pivot element of R_3 equal to one:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & -21 & -10 & 0 \\ 0 & 0 & \boxed{-232/21} & -4 \end{array}\right) \xrightarrow{R_3 \to -(21/232)R_3} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & -21 & -10 & 0 \\ 0 & 0 & 1 & 21/58 \end{array}\right).$$

Next we row reduce all elements above the pivot element of R_3:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & -21 & -10 & 0 \\ 0 & 0 & \boxed{1} & 21/58 \end{array}\right) \xrightarrow[R_2 \to R_2 - 3R_1]{R_2 \to R_2 + 10R_3} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 459/58 \\ 0 & -21 & 0 & 210/58 \\ 0 & 0 & 1 & 21/58 \end{array}\right).$$

Now, we set the pivot element of R_2 to one:

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 459/58 \\ 0 & \boxed{-21} & 0 & 210/58 \\ 0 & 0 & 1 & 21/58 \end{array}\right) \xrightarrow{R_2 \to -1/21R_2} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 459/58 \\ 0 & 1 & 0 & -10/58 \\ 0 & 0 & 1 & 21/58 \end{array}\right).$$

Finally, we row reduce the element above the pivot element of R_2:

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 459/58 \\ 0 & \boxed{1} & 0 & -10/58 \\ 0 & 0 & 1 & 21/58 \end{array}\right) \xrightarrow{R_1 \to R_1 - 2R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 479/58 \\ 0 & 1 & 0 & -10/58 \\ 0 & 0 & 1 & 21/58 \end{array}\right).$$

We can immediately see the equation is consistent because there are no contradictions. The solution is $x = 479/58$, $y = -10/58$, $z = 21/58$. The rank of a matrix is equal to the number of independent equations. Since there are three independent equations, the rank of the matrix is 3.