Let us change

$$V_3 = \begin{bmatrix} 3 \\ 10 \\ 16 \end{bmatrix}$$

Then $m = \begin{bmatrix} 1 & 4 & 9 \\ 2 & 6 & 7 \\ 3 & 10 & 16 \end{bmatrix}$ $\quad m^{-1}$ does not exist $\boxed{\det M = 0}$

By row reduction

$$\sim \begin{bmatrix} 1 & 4 & 9 \\ 2 & 6 & 7 \\ 0 & 0 & 0 \end{bmatrix}$$

rank = 2 < 3

$\therefore$ Linearly dependent

$\boxed{\text{~~Need to put c3=0~~}}$!

$V_3 = V_1 + V_2$

~~Basis~~ $\therefore$ Linearly dependent LD.

$c_1 V_1 + c_2 V_2 + c_3 (V_1 + V_2) = 0$

$\Rightarrow (c_1 + c_3) V_1 + (c_2 + c_3) V_2 = 0$

$\therefore \boxed{c_3 = -c_1 = -c_2}$

* Linear dependence of functions! "Wronskian"

$\boxed{-\frac{d^2 f}{dx^2} = k^2 f}$

$$\begin{cases} f_1(x) = \sin x \\ f_2(x) = \cos x \end{cases}$$

$\boxed{\alpha \stackrel{?}{=} c_1 \sin x + c_2 \cos x = 0}$ $\boxed{\forall x}$

Can $\exists$ find c_1, c_2 st. this eqn is true with non trivial c_1, c_2. ($c_1 = 0 = c_2$ is trivial)

If $\alpha = 0$ $\frac{d}{dx}\alpha = 0$ as well

$$\left.\begin{aligned} \alpha &= c_1 \sin x + c_2 \cos x = 0 \\ \frac{d\alpha}{dx} &= +c_1 \cos x - c_2 \sin x = 0 \end{aligned}\right\} \begin{bmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$\boxed{\det = -1}$

$\therefore c_1 = 0 = c_2$ is only solution.

N functions $(f_1 \ldots f_N)$

$\alpha_1 =$

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_N f_N(x) = 0$$

If we can find non trivial $c_1 \ldots c_N$, then $\exists$ linear dependence.

$$c_1 f_1'(x) + c_2 f_2'(x) + \cdots + c_N f_N'(x) = 0$$

$$\vdots$$

$$c_1 f_1^{(N)}(x) + \cdots + c_N f_N^{(N)}(x) = 0$$

$$W = \det \begin{vmatrix} f_1 & f_2 & \cdots & f_N \\ f_1' & f_2' & & f_N' \\ \vdots & & & \\ f_1^{(N)} & f_2^{(N)} & \cdots & f_N^{(N)} \end{vmatrix}$$

Physics 116A- Winter 2018

Mathematical Methods 116 A

S. Shastry, Feb 21, 2018
Notes on Matrices

A^T_{ij}

A_{ij} $\boxed{(A^T)_{ij} = A_{ji}}$

§Matrix relations

$$\boxed{(A.B)^T = B^T.A^T}$$

$$\boxed{(A.B)_{ij} = \sum_k A_{ik} B_{kj}}$$

LHS $\left((A.B)^T\right)_{ij} = (A.B)_{ji} = \sum_k A_{jk} B_{ki}$

(RHS) $(B^T.A^T)_{ij} = \sum_k (B^T)_{ik} (A^T)_{kj}$

$B_{ki} \quad A_{jk}$

$$\boxed{(B^T)_{ik} = B_{ki}}$$

$$\boxed{(A.B)^{-1} = B^{-1}.A^{-1}}$$

$$\boxed{M.M^{-1} = \mathbb{1} = M^{-1}.M}$$

$(A.B)^{-1}.(A.B) = \mathbb{1}$

$$\boxed{\begin{aligned} A^{-1}.A &= \mathbb{1} = A.A^{-1} \\ B^{-1}.B &= \mathbb{1} = B^{-1}.B \end{aligned}}$$ Known

$$(A.B.C)^{-1} = C^{-1}.B^{-1}.A^{-1}$$

$(B^{-1}.A^{-1}).(A.B) = B^{-1}.\underbrace{A^{-1}.A}.B$
$= B^{-1}.\mathbb{1}.B$
$= B^{-1}.B = \mathbb{1}$
QED

§Trace:

$$Trace A = \sum_{j=1}^{d} A_{jj} \rightarrow Tr A$$

$$Tr(AB) = Tr(BA)$$

$\sum_i (AB)_{ii} \overset{?}{=} \sum_i (B.A)_{ii}$

$\boxed{\sum_i A_{ii} \sum_j B_{jj} = \sum B_{jj} A_{ii}}$

§First introduction to eigenvalues of a matrix and eigenfunctions

$$\boxed{Tr AB = \sum_i (A.B)_{ii} = \sum_{ik} A_{ik} B_{ki}}$$

$$\boxed{Tr(BA) = \sum_{ik} B_{ik} A_{ki}}$$

Let our matrix of interest be called M. Lets keep it simple and choose

$$M = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$$

Suppose we want the solution of a problem

$$M \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

More explicitly, we are asking is there a vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and a number λ such that this relation is true.

Answer: (we will learn how to do this more systematically later)

$$M \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 5 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

and

$$M \begin{bmatrix} -1 \\ 3 \end{bmatrix} = (-2) \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

We will thus say that the eigenvalues are 5 and -2, and that their corresponding eigenvectors are $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ respectively.

If M were the Hamiltonian of a physical system, then the eigenvalues would be the allowed energies of the system, and the eigenfunctions the corresponding wave functions.!!!

More anon as one says, let us get the idea clear that any matrix M has typically n-eigenvalues, where n is the dimension of M.

We may then say that under a suitable procedure (called diagonalization- it is like a rotation of the basis), a matrix M corresponds to a diagonal matrix thus

$$M = \begin{bmatrix} m_{11} & m_{12} \dots & m_{1n} \\ \vdots & \dots & \vdots \\ m_{n1} & m_{n2} \dots & m_{nn} \end{bmatrix} \leftrightarrow \begin{bmatrix} \lambda_1 & 0 \dots & 0 \\ \vdots & \dots & \vdots \\ 0 & 0 \dots & \lambda_n \end{bmatrix}$$

Relating trace and determinant to eigenvalues. Proofs later.

$M = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$ | $Tr\,M = 7$, $det\,M = 10$

$\boxed{\begin{array}{l} \lambda_1 + \lambda_2 = 7 \\ \lambda_1 \times \lambda_2 = 10 \end{array}}$ $\quad \lambda^2 - 7\lambda + 10 = 0$

$(\lambda - \lambda_1)(\lambda - \lambda_2)$

§

$$TrA = \sum_{j=1}^{n} \lambda_j.$$

§

$$DetA = \prod_{j=1}^{n} \lambda_j$$

§Cyclic invariance of trace

$$Tr(ABCD) = Tr(DABC) = Tr(CDAB) \ldots$$

§Determinant formula

$$Det(AB) = Det(A)\; Det(B)$$

$B = \begin{bmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{bmatrix}$ $\quad A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

$A.B = \begin{bmatrix} \lambda_1\mu_1 & 0 & 0 \\ 0 & \lambda_2\mu_2 & 0 \\ 0 & 0 & \lambda_3\mu_3 \end{bmatrix}$

$= (\lambda_1\lambda_2\lambda_3\, \mu_1\mu_2\mu_3)$

$B = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}$

$A.B = \begin{bmatrix} \lambda_1 B_{11} & \lambda_1 B_{12} & \lambda_1 B_{13} \\ \lambda_2 B_{12} & \lambda_2 B_{22} & \lambda_2 B_{23} \\ \lambda_3 B_{31} & \lambda_3 B_{32} & \lambda_3 B_{33} \end{bmatrix}$

$\therefore det(A.B) = (\lambda_1\lambda_2\lambda_3)\, det\, B$

But $det\, A = \lambda_1\lambda_2\lambda_3$

$\therefore \boxed{QED}$

§Operations on matrices and some Special matrices important in Physics applications

- CC A^*
- Transpose A^T $\quad (A^T)_{ij} = A_{ji}$
- Orthogonal $A^T.A = \mathbf{1}$
- Hermitean Adjoint $A^\dagger$ $\quad A^\dagger = (A^*)^T \quad \boxed{(A^\dagger)_{ij} = A^*_{ji}}$ $\quad \boxed{A^\dagger = A}$
- Real symmetric

$$A = A^* = A^T$$

- Hermitean

$$A^\dagger = A$$

- Unitary

$$\boxed{A^\dagger = A^{-1}}$$

- Commuting matrices

$$A.B = B.A$$

$\mathrm{Tr}\, M = 7$

$\det M = 10$

$M = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$

- Non-commuting matrices

$$A.B \neq B.A$$

$\boxed{\begin{matrix} \lambda_1 + \lambda_2 = 7 \\ \lambda_1 \times \lambda_2 = 10 \end{matrix}}$

$\lambda^2 - 7\lambda + 10 = 0$

$(\lambda - \lambda_1)(\lambda - \lambda_2)$

- Diagonal matrix (*not the identity matrix*)
- If A and B commute, they are simultaneously diagonalizable.

$\begin{bmatrix} \lambda_1 & 0 & \\ & \lambda_2 & \\ & 0 & \lambda_3 \end{bmatrix}$

§First example of diagonalization
Take our old problem

$\lambda_+ = 5$

$\begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$\boxed{x_1 = 2x_2}$

$\boxed{x_1/x_2 = 2}$

$\boxed{x_1^2 + x_2^2 = 1}$

$$M = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$$

and

$$M \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\begin{bmatrix} 4x_1 + 2x_2 \\ 3x_1 - x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\boxed{\begin{matrix} 4x_1 + 2x_2 = \lambda x_1 \\ 3x_1 - x_2 = \lambda x_2 \end{matrix}} \Rightarrow \begin{bmatrix} 4-\lambda & 2 \\ 3 & -\lambda-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$\boxed{\det \begin{vmatrix} 4-\lambda & 2 \\ 3 & -\lambda-1 \end{vmatrix} = 0}$

$(4-\lambda)(1+\lambda) + 6 = 0$

$-\lambda^2 + 3\lambda + 10 = 0$

$\boxed{\lambda^2 - 3\lambda - 10 = 0}$

$\lambda = \frac{3 \pm \sqrt{9+40}}{2} = \frac{3 \pm 7}{2} \left\{ \begin{matrix} \lambda_+ = 5 \\ \lambda_- = -2 \end{matrix} \right.$

§Second example: From geometry:

$$ax^2 + by^2 + 2cxy = 1 \tag{1}$$