Physics 116A- Winter 2018

Mathematical Methods 116 A

S. Shastry, March 1st, 2018
Notes on Matrices

Recall that we calculated the determinant

$$\Delta(\lambda) \equiv det\big|M - \lambda \mathbb{1}\big| \tag{1}$$

which is a polynomial in λ of degree d where d is the dimension of the matrix M. The eigenvalues are the roots of $\Delta(\lambda) = 0$. This object $\Delta(\lambda)$ is called the *characteristic polynomial* of the matrix M.

Recall a simple theorem from determinants that says

$$detA = detA^T \tag{2}$$

This implies that

§**Comment: The eigenvalues of M and M^T are the same!!**

We saw that a generic i.e. non-symmetric (or non Hermitean) matrix $M \neq M^T$ has eigenvalues given by the condition

$M.\psi_j = \lambda_j \psi_j,$
and hence taking transpose of both sides
$\psi_j^T.M^T = \lambda_j \psi_j^T.$

This tells us that M^T has "eigenvectors" ψ_j^T with the same eigenvalues λ_j. This is consistent with above comment, but brings in a new point: what is ψ^T doing to the left of M^T.

Question: Could we also have a similar relation for M, rather than M^T with something to its left?

§**Left and right eigenfunctions**

Answer is yes!!

A non-symmetric matrix M can have for each eigenvalue, a distinct right and left eigenvector.

Thus

$$M.\psi_R = \lambda\psi_R, \text{ and also with same eigenvalue } \psi_L^T.M = \lambda\psi_L^T$$

In particular

$$\psi_L \neq \psi_R. \tag{3}$$

§**Re-Return to first example of diagonalization**

In our old problem

$$M = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$$

we saw that there are two eigenvalues $\lambda = \lambda_{a,b}$ where $\lambda_a = 5$ and $\lambda_b = -2$ and there are two corresponding **right** eigenvectors, ψ_a and ψ_b given by

$$\psi_{R,a} = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}, \text{and } \ \psi_{R,b} = \begin{bmatrix} \frac{1}{\sqrt{10}} \\ \frac{-3}{\sqrt{10}} \end{bmatrix}, \tag{4}$$

such that

$$M.\psi_{R,a} = \lambda_a \psi_{R,a}, \ \ M.\psi_{R,b} = \lambda_b \psi_{R,b}. \tag{5}$$

Question: How about left eigenvectors?

$$\begin{bmatrix} a & b \end{bmatrix} . \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} = \lambda \begin{bmatrix} a & b \end{bmatrix}. \tag{6}$$

This gives us the pair of linear eqns.

$$(4-\lambda)a + 3b = 0, \ \ 2a - b(1+\lambda) = 0 \tag{7}$$

The characteristic determinant

$$\Delta(\lambda) = 10 + 3\lambda - \lambda^2, \tag{8}$$

with roots $\lambda = 5, -2$. Hence the eigenvalues are the same as those from the right operation.

How about left eigenvectors?

$$\psi_{L,a}^T = \begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{bmatrix}, \text{ and } \psi_{L,b}^T = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \end{bmatrix} \tag{9}$$

Comparing Eq. (9) with Eq. (4)- close but not the same!!

$$\psi_L \neq \psi_R. \tag{10}$$

§ **Powers of matrices and matrix relations made easy!**

We found the diagonal matrix D by computing the "similarity transformation"

$$W^{-1}.M.W = D \tag{11}$$

Recall that

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & \dots \\ 0 & \lambda_2 & 0 & \dots \\ \vdots & \vdots & \vdots & \\ 0 & \dots & 0 & \lambda_n \end{bmatrix} \tag{12}$$

Therefore

$$D^m = \begin{bmatrix} \lambda_1^m & 0 & 0 & \dots \\ 0 & \lambda_2^m & 0 & \dots \\ \vdots & \vdots & \vdots & \\ 0 & \dots & 0 & \lambda_n^m \end{bmatrix} \tag{13}$$

Now take the mth power of the left hand side of Eq. (15)

$$(W^{-1}.M.W).(W^{-1}.M.W).(W^{-1}.M.W)\dots(W^{-1}.M.W) = W^{-1}.M^m.W \tag{14}$$

Hence we need to calculate the W only once, and this gives us all the powers of the matrix M in a diagonal form through

$$W^{-1}.M^m.W = D^m \tag{15}$$

We proved:

$$W^{-1}.M^m.W = D^m$$

We can use this for many important identities.

1. $TrM = \sum_{j=1}^{n} \lambda_j$
2. $DetM = \prod_{j=1}^{n} \lambda_j$.
3. $DetM = e^{Tr \log M}$

§**Cayley-Hamilton theorem**

Let us recall that the eigenvalues are the roots of the characteristic polynomial,

$$\Delta(\lambda) = \lambda^n + A_1 \lambda^{n-1} + \ldots + A_{n-1} = 0 \tag{16}$$

where A_j can be calculated given the matrix M.

Cayley-Hamilton argue that the matrix also satisfies the condition

$$M^n + A_1 M^{n-1} + \ldots + A_{n-1} \mathbb{1} = 0 \tag{17}$$

Proof is immediate, upon using Eq. (15).