Physics 116A- Winter 2018

Mathematical Methods 116 A

S. Shastry, March 13, 2018

Notes on Linear vector spaces

§ **Linear Vector Spaces (LVS) come in many forms, often surprising ones**

What is a linear vector space?

A collection of vectors $\{V_j\}$ with $1 \leq j \leq N$ is called a LVS $\mathcal{W}$ if they satisfy:

- Commutative Addition $V_i + V_j = V_j + V_i$
- Associative $V_i + (V_j + V_k) = (V_i + V_j) + V_k$
- Multiplication by a scalar: $aV_j \in \mathcal{W}$
- Multiplication by identity: $1 \times V_j = V_j$
- Existence of inverse under addition $V_j = \emptyset - V_j$

Standard examples: Vectors in d-dimensions are familiar to us

Non-standard example: *Polynomials of degree m*

Consider expressions of type:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_m x^m.$$

They form a LVS. (Checked easily. Consider other polynomials of same type with different coefficients a'_j)

§**Basis**

LVS's have a basis, e.g. in 3-d the vectors $\hat{x}, \hat{y}, \hat{z}$.

How about a basis for polynomials?

Consider $m = 3$

$$\{\phi_0(x) = 1, \phi_1(x) = x, \phi_2(x) = x^2, \phi_3(x) = x^3\}$$

Any polynomial of degree 3 is a linear combination of these. Linearly independent as we can see by forming the Wronskian (determinant)

$$W = \begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & 2x & 3x^2 \\ 0 & 0 & 2 & 6x \\ 0 & 0 & 0 & 6 \end{vmatrix} = 12 \tag{1}$$

Since $W \neq 0$, these are linearly independent.

§Overlap, length, orthonormality of polynomials

We introduce an inner product

$$\phi_i^T.\phi_j = \int_{-1}^{1} dx \phi_i(x)\phi_j(x)$$

Are they orthogonal? No: Are they normalized?
We can call in Gram-Schmidt

- $\psi_0 = \frac{1}{\sqrt{2}}\phi_0$, clearly $\int_{-1}^{1} dx\ \psi_0^2 = 1$
- $\psi_1 = a_0(\phi_1 - b_0\psi_0)$.
- This requires $b_0 = \int_{-1}^{1}\ dx\ \phi_1(x)\psi_0 = \frac{1}{\sqrt{2}}\frac{1}{2}x^2\Big|_{-1}^{1} = 0$.
- For normalization we need $1 = a_0^2 \int_{-1}^{1}\ dx\ x^2 = \frac{2}{3}a_0^2$. Hence $a_0 = \sqrt{2/3}$

The book has further terms worked out. This is the genesis of the Legendre polynomials. Read MB

§**General inner products and norms**

Let $A_n(x)$ be some sequence of functions

$$\text{Inner product}[A_i(x), A_j(x)] \equiv \int_a^b dx\, A^*(x)B(x)$$

$$\text{Norm}||A(x)|| \equiv \int_a^b dx\, A^*(x)A(x) \geq 0$$

§**Dirac notation**

This paves the way for the Dirac notation.

- $< A|B >= \int A^*(x)B(x)\, dx$
- $< A|A >\geq 0$. Equals 0 only if $A = 0$
- $< C|aA + bB >= a < C|A > +b < C|B >$.
- Schwartz inequality
- Triangle inequality