Physics 116A- Winter 2018

Mathematical Methods 116 A

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Notes on Matrices

§ **Powers of matrices and matrix relations made easy!**

We found the diagonal matrix D by computing the "similarity transformation"

$$W^{-1}.M.W = D \tag{1}$$

Recall that

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & \dots \\ 0 & \lambda_2 & 0 & \dots \\ \vdots & \vdots & \vdots & \\ 0 & \dots & 0 & \lambda_n \end{bmatrix} \tag{2}$$

§**Cayley-Hamilton theorem**

Let us recall that the eigenvalues are the roots of the characteristic polynomial,

$$\begin{aligned} \Delta(\lambda) &= \lambda^n + A_1\lambda^{n-1} + \ldots + A_{n-1} = 0 \\ &= \prod_{j=1}^{n}(\lambda - \lambda_j) \end{aligned} \tag{3}$$

where A_j can be calculated given the matrix M and yield the n eigenvalues λ_j.

Cayley-Hamilton argue that the matrix also satisfies the condition

$$M^n + A_1 M^{n-1} + \ldots + A_{n-1}\mathbb{1} = 0 \tag{4}$$

How does one prove such a theorem?

Call

$$\mathcal{L} = M^n + A_1 M^{n-1} + \ldots + A_{n-1}\mathbb{1} \tag{5}$$

and observe

$$W^{-1}.\mathcal{L}.W = D^n + A_1 D^{n-1} + \ldots + A_{n-1}\mathbb{1}. \tag{6}$$

Proof

Therefore we take the explicit form of D into account and hence Eq. (6) is a diagonal matrix. Take the j^{th} element of that hence

$$\begin{aligned} \left(W^{-1}.\mathcal{L}.W\right)_{jj} &= \lambda_j^n + A_1\lambda_j^{n-1} + \ldots + A_{n-1} \\ &= 0. \text{ on using Eq. (3)} \end{aligned} \tag{7}$$

§**Important properties of Hermitean matrices**

We will show a few important properties of Hermitean matrices, i.e. matrices $H = H^{\dagger}$. Recall

$$(H^{\dagger})_{ij} = (H_{ji})^*,$$

and these are of great importance in QM. Also recall that real symmetric matrices are special cases if Hermitean.

§**Eigenvalues of Hermitean matrices are real**

Let us consider the eigenvalue,

$$H\psi_j = \lambda_j\psi_j \tag{8}$$

so we can take the Hermitean adjoint of this equation and get

$$\psi_j^{\dagger}H^{\dagger} = \psi_j^{\dagger}H = \psi_j^{\dagger}\lambda_j^*. \tag{9}$$

Keep in mind that $\psi^{\dagger}$ is a row-vector found by taking the complex conjugate and transpose of ψ_j.

Now using both these equations we get

$$\begin{aligned} \psi_j^{\dagger}.H.\psi_j &= \lambda_j\psi_j^{\dagger}.\psi_j \\ &= \lambda_j^*\psi_j^{\dagger}.\psi_j \end{aligned} \tag{10}$$

Hence

$$\lambda_j = \lambda_j^*. \tag{11}$$

§**Eigenvectors of Hermitean matrices with distinct eigenvalues are orthogonal to each other.**

Suppose we have two eigenvalues and their eigenvectors as

$$\begin{aligned} H.\psi_i &= \lambda_i\psi_i \\ H.\psi_j &= \lambda_j\psi_j \end{aligned} \tag{12}$$

with

$$\lambda_i \neq \lambda_j \tag{13}$$

We can show that

$$\psi_i^\dagger . \psi_j = 0. \tag{14}$$

Proof: Consider

$$\begin{aligned} \psi_i^\dagger . H . \psi_j & ? \\ & = \lambda_i \, \psi_i^\dagger . \psi_j \\ & = \lambda_j \, \psi_i^\dagger . \psi_j \end{aligned} \tag{15}$$

hence the result.

§**Degeneracy**

An eigenvalue is called degenerate if we can find two independent eigenvectors for the same eigenvalue. This implies basically that the characteristic polynomial has repeated roots.

For Hermitean matrices, we can find as many eigenvectors as the dimension of matrices, degenerate or otherwise. Gram-Schmit is useful if degeneracy is found.

§**Re-Return to first example of diagonalization**

In our old problem

$$M = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$$

we saw that there are two eigenvalues $\lambda = \lambda_{a,b}$ where $\lambda_a = 5$ and $\lambda_b = -2$ and there are two corresponding **right** eigenvectors, ψ_a and ψ_b given by

$$\psi_{R,a} = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}, \text{and } \ \psi_{R,b} = \begin{bmatrix} \frac{1}{\sqrt{10}} \\ \frac{-3}{\sqrt{10}} \end{bmatrix}, \tag{16}$$

Also we found two left eigenvectors.

$$\psi^T_{L,a} = \begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{bmatrix}, \text{ and } \psi^T_{L,b} = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \end{bmatrix} \tag{17}$$

We commented: Comparing Eq. (17) with Eq. (16)- close but not the same!!

$$\psi_L \neq \psi_R. \tag{18}$$

Can we say something more here about orthogonality?

Consider

$$\begin{aligned} \psi^T_{L,a}.M.\psi_{R,b} &= \\ \psi^T_{L,b}.M.\psi_{R,a} &= \end{aligned}$$

§**One example of diagonalizing a Hermitean matrix**
Consider

$$M = \begin{bmatrix} 2 & 2i \\ -2i & -1 \end{bmatrix}$$

Eigenvalues : $\lambda_1 = 3, \lambda_2 = -2.$

Eigenvectors.

$$\psi_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2i \\ 1 \end{bmatrix}, \; \psi_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -i \\ 2 \end{bmatrix} \tag{19}$$

Matrix of eigenvectors. $U = [\psi_1, \psi_2]$

§Application of matrix diagonalization to physical problems

Consider three coupled particles

$$H = \frac{m}{2}\left((\dot{u}_1)^2 + (\dot{u}_2)^2 + (\dot{u}_3)^2\right) + \frac{k}{2}\left((u_1 - u_2)^2 + (u_3 - u_2)^2\right) \tag{20}$$

Assume harmonic motion, i.e. $u_1 = ae^{i\omega t}$, $u_2 = be^{i\omega t}$, $u_3 = ce^{i\omega t}$, where a, b, c are three amplitudes of oscillation.

EOM:

$$\begin{aligned} m\omega^2 a - k(a-b) &= 0 \\ m\omega^2 b - k(2b - a - c) &= 0 \\ m\omega^2 c - k(c-b) &= 0 \end{aligned} \tag{21}$$

we may define

$$\Lambda = \frac{m\omega^2}{k}, \tag{22}$$

and rearrange this as a matrix equation

$$\begin{bmatrix} \Lambda - 1 & 1 & 0 \\ 1 & \Lambda - 2 & 1 \\ 0 & 1 & \Lambda - 1 \end{bmatrix} . \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0 \tag{23}$$

This is our favorite matrix equation and hence we can turn the crank. The eigenvalues written in terms of Λ are

$$\Lambda = 3, 1, 0 \tag{24}$$

and the un-normalized eigenfunctions are

$$\psi_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \ \psi_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \ \psi_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{25}$$