Physics 116A- Winter 2018

Mathematical Methods 116 A

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Notes on Complex numbers

Origin of complex numbers We mentioned the fundamental theorem of algebra. Any polynomial of degree n has exactly n roots. For example

$$f(x) = x^2 + x + 2 \tag{1}$$

can be easily seen to have two roots $x = 1, -2$ so that $f(x) = (x-1)(x+2)$. What about the case

$$f(x) = x^2 + 4, \tag{2}$$

the two roots are non-real. In fact the two roots are $x = \pm 2i$ where $i = \sqrt{-1}$. In EE the variable $i \to j$ since i is reserved for currents. We will use the standard notation i here.

Decomposing complex numbers A complex number is associated with its complex conjugate z^* so that $(z^*)^* = z$. We may write a complex number z in one of many convenient ways

$$\begin{aligned} z &= x + iy, \text{ Real and Imaginary parts visualized in 2-dimensions} \\ &= re^{i\theta}, \text{ Polar decomposition- again 2-dimensions.} \end{aligned} \tag{3}$$

The ranges are $-\infty \leq x, y \leq \infty$ and $0 \leq r \leq \infty$ and $0 < \theta \leq 2\pi$. Note that with m an integer, θ and $\theta + 2m\pi$ lead to the same z and hence are non-distinct.

These two decompositions can be defined as

$$\begin{aligned} x &= \frac{(z + z^*)}{2} = \Re e z, \\ y &= \frac{(z - z^*)}{2i} = \Im m z \end{aligned}$$

and

$$\begin{aligned} r &= \sqrt{zz^*} = \sqrt{x^2 + y^2} \\ \tan(\theta) &= \frac{y}{x}. \end{aligned} \tag{4}$$

Here the angle θ is in radians.

Recall conversion π (Radians) $= 360^0$(Degrees). In calculus the variables such as x in $\sin(x)$ are assumed to be in radians.

§**Operations with complex numbers.**

- Visualize complex plane: i.e. locate z in the 2-d complex plane.
- Conversion from polar to other (x,y) form
- Add $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$.
- Multiply $z_1 z_2$
- Complex conjugate z^*
- Divide z_1/z_2
- Euler's equation
$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$
- de Moivre's formula
$$e^{ip\theta} = \cos(p\theta) + i\sin(p\theta).$$
- Powers z^n
- Roots. N roots of $z^N = r$. Here $r > 0$ is a real number say 1.
$$z^N = re^{i2\pi j}$$
$$z = r^{1/N} e^{i2\pi j/N},\ j = 0, 1, 2, \ldots (N-1),$$
since increasing $j \to j + N$ leads to same number.
Example
$$z^3 = 8,$$
has 3 solutions
$$z = 2\{1, e^{i2\pi/3}, e^{i4\pi/3}\}.$$
These are written explicitly using de Moivre's theorem as
$$z = 2\{1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2}\}.$$