Physics 116A- Winter 2018

Mathematical Methods 116 A

S. Shastry, January 30, 2018

Notes on Complex Variables and Physical Applications

§**LCR Circuits** We will consider a neat application of complex numbers

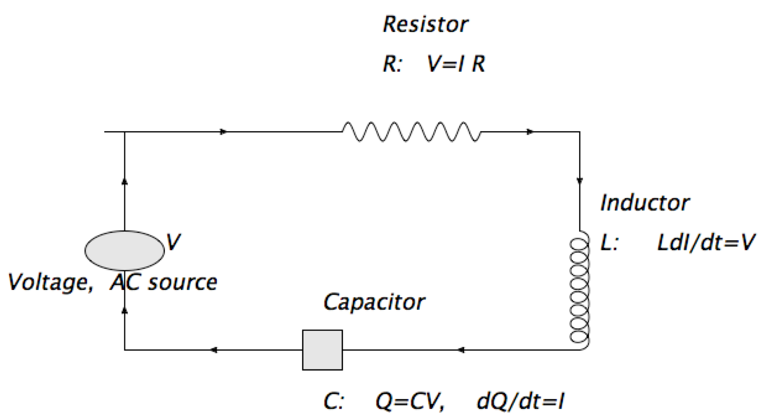

Figure 1: The three elements of the circuit are (1) a resistor R (2) an inductance I and (3) a capacitor C. The voltage drop across each of them satisfies $V_1 = IR$, $LdI/dt = V_2$, and $Q = CV_3$,. The total voltage drop $V = V_1 + V_2 + V_3$ giving us the equation Eq. (3) to solve.

which is of central importance in electrical circuits. An alternating electrical current and voltage are encountered in most real-life applications including home use of electricity. We note that the current and voltage are periodic functions of time which can be written most generally as

$$
\begin{aligned}
V(t) &= V_0\cos(\omega t+\phi_1) = \frac{V_0}{2}(e^{i(\omega t+\phi_1)} + e^{-i(\omega t+\phi_1)}) \\
I(t) &= I_0\cos(\omega t+\phi_2) = \frac{I_0}{2}(e^{i(\omega t+\phi_2)} + e^{-i(\omega t+\phi_2)}), \qquad (1)
\end{aligned}
$$

where ω is the angular frequency (for e.g. 60 cycles per second in household supply in the US), and V_0 and I_0 are the amplitudes of the two variables, and ϕ_1, ϕ_2 are two time independent phases that are needed for describing the most general situation. Let us imagine a circuit, as in the figure, with a resistor R an inductor I and a capacitor C. Considerations of physics discussed in Physics 5, give us the equations

$$V_1 = IR, \;\; V_2 = LdI/dt, \;\; \text{and} \;\; V_3 = Q/C, \tag{2}$$

where V_j is understood as the potential drop across the resistor, inductor and capacitor. The charge $Q(t) = \int_0^t I(t')dt'$ so that $dQ/dt = I$. The total voltage drop of the source matches the sum over the three elements so that

$$\begin{aligned} V(t) &= V_1 + V_2 + V_3 \\ &= I(t)R + LdI(t)/dt + 1/C \int_0^t dt' \, I(t'). \end{aligned} \tag{3}$$

This is the differential equation that we need to solve, we will see that it is easily done using complex variables.

Let us first note that the physical currents and voltages in Eq. (1) are real. That is because we can measure them using (real) voltmeters and (real) ammeters, and they give us a real shock, if they are high enough!!

§**Solution using real variables only** We can solve Eq. (1) if we assume $I = I_0 \cos(\omega t)$ (i.e. setting $\phi_2 = 0$ in Eq. (1)) using only real variables. This can be carried out and gives (check this)

$$V(t) = I_0 \left(R \, \cos(\omega t) + \sin(\omega t)[\frac{1}{C\omega} - L\omega] \right) \tag{4}$$

While this is correct, it is still not very useful, since it remains very far from a simple Ohm's law form. In particular the time dependence is not simple at all, it is a mix of sine and cosines.

We can march on and iron this out using the addition theorem for cosine

$$\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B), \tag{5}$$

and plug in $R = |Z|\cos(\theta)$ and $L\omega - 1/(C\omega) = |Z|\sin(\theta)$ in terms of two unknowns $|Z|, \theta$. We can solve for these immediately so that

$$\begin{aligned} |Z| &= \sqrt{R^2 + (L\omega - 1/(C\omega))^2} \\ \tan\theta &= \frac{(L\omega - 1/(C\omega))}{R}. \end{aligned} \tag{6}$$

With this we write

$$V(t) = I_0|Z|\cos(\omega t + \theta). \tag{7}$$

The physical voltage is not simply $\cos(\omega t)$ as in the current, but rather, has a "phase shift" θ relative to the current.

§**Solution using complex variables** We now see that a solution can be found by assuming that

$$I = I_0 e^{i\omega t}. \tag{8}$$

We could plug this into Eq. (3) and find V as

$$V(t) = I_0 e^{i\omega t} \left(R + i(L\omega - \frac{C}{\omega}) \right) + iI_0 \frac{C}{\omega}, \tag{9}$$

where we used the simple relations

$$\begin{aligned} de^{i\omega t}/dt &= i\omega e^{i\omega t} \\ \int_0^t dt' e^{i\omega t'} &= -\frac{i}{\omega}(e^{i\omega t} - 1). \end{aligned} \tag{10}$$

You may feel queasy here since $V(t)$ is complex, not real. We can fix this. For this purpose, let us imagine we do a similar calculation with

$$I = I_0 e^{-i\omega t}, \tag{11}$$

the negative frequency version of the above calculation. Since $\omega \to -\omega$, this is equivalent to taking the complex conjugate. Hence we will be able to write down the answer immediately from taking complex conjugate of Eq. (9) and write the answer as

$$V^*(t) = I_0 e^{-i\omega t} \left(R - i(L\omega - \frac{C}{\omega}) \right) - iI_0 \frac{C}{\omega}. \tag{12}$$

Hence the final answer for the problem of applying $I = I_0 \cos(\omega t)$ is real, it is

$$V_{phys}(t) = (V(t) + V^*(t))/2 \tag{13}$$

i.e. obtained by taking the real part of the solution Eq. (9).

Notice that the annoying time independent last term in Eq. (9) and Eq. (12) cancels out. Hence in most practical calculations, one simply drops the constant of integration in Eq. (10).

Let us verify that we indeed get the correct answer. Let us take the real part of Eq. (9)

$$V_{phys} = \Re e\, V(t) = I_0\, \Re e\, e^{i\omega t} \times \left(R + i(L\omega - \frac{C}{\omega}) \right). \tag{14}$$

We define a complex ω dependent impedance Z through

$$Z(\omega) = \left(R + i(L\omega - \frac{C}{\omega}) \right), \tag{15}$$

and hence

$$V_{phys}(t) = \Re e\, Z(\omega) I_0 e^{i\omega t} = \Re e\, Z(\omega) I(t). \tag{16}$$

We can push further using the polar decomposition of the complex impedance

$$Z = |Z|e^{i\theta} \tag{17}$$

where $|Z|$ and θ are identical to the ones we calculated earlier in Eq. (6), and hence recover the result from Eq. (7)

$$V_{phys} = |Z(\omega)| I_0 \cos(\omega t + \theta). \tag{18}$$

§**Comments** (a)The complex treatment of LCR circuits is universally used. The EE folks use a different notation for imaginary numbers, they use $i \to j$ since i means the current to most of them.

(b)Another comment is the case of resonance, the LCR circuit is said to be in resonance if the impedance Z is purely real. This requires

$$\begin{aligned} L\omega_R &= 1/(C\omega_R) \\ \omega_R &= \sqrt{LC}, \end{aligned} \tag{19}$$

and thus given a L and a C, there is a unique resonance frequency ω_R given by Eq. (19).

(c) The summary of the complex variable method for solving Eq. (3) is as follows.

- We assume that I and V have an identical *complex* time dependence, so $I = I_0 e^{i\omega t}$ and $V = V_0 e^{i\omega t}$
- In Eq. (3) we replace $d/dt \to i\omega$ and $\int dt \to \frac{1}{i\omega}$ and write it out Eq. (3) as an algebraic equation

$$Z(\omega) = V_0/I_0 = \left(R + i(\omega R - \frac{1}{\omega C}) \right) \tag{20}$$

  This is the complex version of Ohm's law.
- Writing $Z = |Z|e^{i\theta}$ gives us the phase shift and the absolute value of the impedance.

- This procedure can be implemented across *any* configuration of L,C, R elements in a complicated circuit.

§**Harmonic oscillators with damping.**
(Damping is not given in Boas's book. Therefore please note this discussion carefully.)

$$m\frac{d^2x}{dt^2} = -\Gamma\frac{dx}{dt} - kx, \tag{21}$$

where the middle term is called a damping term, since $v = \frac{dx}{dt}$ is the velocity. This damping term is beyond Newton's laws, and represents the frictional forces due to the environment.

If $\Gamma = 0$ then we can solve the equation easily as $x(t) = x_0\, cos(\omega t + \theta)$ where $\omega = \sqrt{k/m}$ and θ is arbitrary. We can also solve it using complex variables:

$$x(t) = x_0 e^{i\omega t+\theta} \tag{22}$$

with $x_{phys} = \Re e\, x(t)$ as in the LCR problem. This also gives $\omega = \sqrt{k/m}$ and θ is arbitrary.

Let us see what happens when a small positive $\Gamma \neq 0$, i.e. we have a small amount of damping, using the complex variable subsitution. Now Eq. (21) becomes

$$-m\omega^2 x_0 = -i\omega\Gamma x_0 - kx_0, \tag{23}$$

and hence

$$\omega(\omega - i\Gamma) = k/m \tag{24}$$

We can solve this quadratic and get

$$\omega = i(\frac{\Gamma}{2m}) + \sqrt{\frac{k}{m} - (\frac{\Gamma}{2m})^2}. \tag{25}$$

We chose the plus sign in the quadratic solution, since at $\Gamma = 0$ this should give us back $\omega = \sqrt{k/m}$. If we assume small Γ we can approximate this by

$$\omega = i(\frac{\Gamma}{2m}) + \sqrt{\frac{k}{m}}, \tag{26}$$

and hence write the solution

$$x(t) = x_0 \Re e\, e^{i\sqrt{k/m}\,t - (\frac{\Gamma}{2m})t} = x_0 \cos(\sqrt{\frac{k}{m}}\,t) \times e^{-(\frac{\Gamma}{2m})t}, \tag{27}$$

and hence $x(t)$ not only oscillates as $\cos(\sqrt{k/m}\,t))$, it also decays in time like $e^{-t/\tau}$ where

$$\tau = \frac{2m}{\Gamma}. \tag{28}$$

This decay is what we expect from damping.