Physics 116A- Winter 2018

Mathematical Methods 116 A

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Notes on Linear Equations

§**Simultaneous linear equations in terms of matrices.**

Suppose we want to solve a pair of linear equations in two variables

$$\begin{aligned} 4x_1 + 2x_2 &= 1 \\ 3x_1 - x_2 &= 4 \end{aligned} \tag{1}$$

This can be solved easily: we get $x_1 = 9/10$ and $x_2 = -13/10$. We can also solve it as a matrix problem, by a method that generalizes to much more complex situations where automation is needed. We rewrite Eq. (1) as a matrix problem

$$\begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}. \tag{2}$$

We could also write Eq. (2) for N simultaneous equations as

$$\sum_{j=1}^{N} M_{ij} x_j = k_i, \;\; \text{for} \;\; i = 1, 2, \ldots N. \tag{3}$$

This can be written more abstractly in matrix notation as

$$M.x = k \tag{4}$$

where the $N \times N$ matrix M is given as

$$M = \begin{bmatrix} M_{11} & M_{12} & \ldots & M_{1N} \\ M_{21} & M_{22} & \ldots & M_{2N} \\ \vdots & \vdots & \vdots & \\ M_{N1} & M_{N2} & \ldots & M_{NN} \end{bmatrix}, \;\; x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}, \;\; k = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_N \end{bmatrix}. \tag{5}$$

You will notice that $N = 2$ gives back Eq. (2). We said that M is a $N \times N$ matrix with N rows and N columns. The vector x can also be viewed as a matrix, but now a $N \times 1$ matrix with N rows and 1 column.

§**The matrix notation to remember**.

For any $N \times M$ matrix A,

$$A_{ij} = \text{Entry in i}^{th} \text{ row j}^{th} \text{ column}$$

$$\begin{bmatrix} & Col\text{-}1 & Col\text{-}2 & \dots & Col\text{-}M \\ Row\text{-}1 & A_{11} & A_{12} & \dots & A_{1M} \\ Row\text{-}2 & A_{21} & A_{22} & \dots & A_{2M} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ Row\text{-}N & A_{N1} & A_{N2} & \dots & A_{NM} \end{bmatrix} \tag{6}$$

Getting back to Eq. (5) we notice that the numerical entries are in M and k, so that writing x is the same for all problems of this class, and is almost redundant.

In order to present the linear equations problem in the most compact way we can as well add k's to the matrix and define a $N \times (N+1)$ matrix called the augmented matrix,

$$A = \begin{bmatrix} M_{11} & \dots & M_{1N} & k_1 \\ M_{21} & \dots & M_{2N} & k_2 \\ \vdots & \vdots & \vdots & \vdots \\ M_{N1} & \dots & M_{NN} & k_N \end{bmatrix} \tag{7}$$

Keep in mind the original M matrix, and when we add the x to it, we get the augmented matrix A.

§**Operations with the augmented matrix:**

We want to simplify an augmented matrix to make as many elements vanish as possible, by a procedure called *row reduction*

To simplify an augmented matrix by *row reduction*

1) We can multiply any row with a common number, 2)We can add or subtract any two rows. 3) Exchange any two rows.

§**Rank of a general matrix**

It is the number of non-zero rows after row reduction

More formally

Given any $m \times N$ matrix A, we can proceed by the given row reduction rules to generate a matrix $\widetilde{A}$, which cannot be reduced further. **The rank of A is defined as the number of non-zero rows of $\widetilde{A}$.**

Example: (A)

$$A = \begin{bmatrix} 1 & 3 & 4 & 5 \\ 2 & 1 & 5 & 6 \\ 3 & 1 & 11 & 7 \end{bmatrix} \rightarrow \widetilde{A} = \begin{bmatrix} 1 & 0 & 0 & \frac{67}{19} \\ 0 & 1 & 0 & \frac{20}{19} \\ 0 & 0 & 1 & -\frac{8}{19} \end{bmatrix}$$

We did the row reduction step-by-step in class.

Both matrices A and M have three non zero rows at the end of the row reduction. Hence we have found Rank (M)= 3, Rank (A)= 3.

Consistent, i.e. we can solve for the three variables x_1, x_2, x_3 uniquely, as $67/19, 20/19, -8/19$.

Example: (B)

$$A = \begin{bmatrix} 1 & 3 & 4 & 5 \\ 2 & 1 & 5 & 6 \\ 3 & 4 & 9 & 7 \end{bmatrix} \rightarrow \widetilde{A} = \begin{bmatrix} 1 & 3 & 4 & 5 \\ 0 & 1 & 3/5 & 4/5 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Here we see that Rank (M)= 2 and Rank (A)= 3.

§**Consistency rules** These can be applied after the ranks of the matrices A and M are calculated. The **three rules** are

1. If Rank(M)=Rank(A)= N, where N is the number of variables, then we have a consistent solution for all N variables.

2. If Rank(M)=Rank(A)= N', where $N' < N$ is less than the number of variables, then we have a consistent solution for N' variables, and the remaining N-N' variables are undetermined.

3. If Rank(M) $<$ Rank(A), then the equations are inconsistent (i.e. make no sense).

Our example A is a case where Rule # 1 is applicable.

Example B is a case where Rule #3 is applicable.

In the HW # 3 we will find examples where the Rule #2 is applicable.

§**Determinants**

Given a $N \times N$ matrix M we can define its determinant $detM = |M|$ for increasing N as follows.

§N=1

$$DetM = M$$

§N=2

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \; Det\, M = ad - bc.$$

§N=3

$$M = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},\ Det\, M = a_{11}C_{11} - a_{12}C_{12} + a_{13}C_{13}, \qquad (8)$$

where the *co-factors* C_{ij} are given by

$$C_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix},\ C_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix},\ C_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix},$$

This is an example of Laplace's expansion of a matrix in terms of co-factors. The co-factor C_{nm} is defined as a sign (plus or minus) into the determinant of the matrix M_{nm}, found by canceling the row and column of the original matrix that intersect at n, m. M_{nm} is called the m,n th minor of the matrix M. The sign in the Laplace expansion alternates with the column number.

It is important to note that Laplace's expansion can be done in more than one ways, they all lead to the same final answer. For example, we can expand M along the first column and get

$$DetM = a_{11}C_{11} - a_{21}C_{21} + a_{31}C_{31},$$

where $C's$ follow the same definitions.

We could also expand using the second column, etc.

These alternatives follow from the invariance of the determinant:

Properties and Invariances of the determinant

- The determinant is zero if any two rows coincide, or any two columns coincide. Also true for proportional instead of coincide.
- The determinant is unchanged by adding a row to all other rows, or a column to all other columns.
- We may multiply a determinant by a constant which can be distributed over any single row or a single column. Thus for example

$$c \times \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} c\,a_{11} & a_{12} & a_{13} \\ c\,a_{21} & a_{22} & a_{23} \\ c\,a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ c\,a_{21} & c\,a_{22} & c\,a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \qquad (9)$$

§Examples

§**Cramer's rule**

§**Rank of a matrix redefined via determinants.**

§**Vectors: cross products and determinants**