Physics 116A- Winter 2018

Mathematical Methods 116 A

S. Shastry, Jan, 10 2018

Notes on power series and series expansion

Power series: A power series is analogous to the earlier discussed series but has a new dimension to it, it involves the powers of a variable "x". Thus

$$P(x) = \sum_{n=0}^{\infty} a_n \, x^n, \tag{1}$$

is a power series in x. If the sum is truncated at some finite order M, we would get a polynomial of degree M

$$P_M(x) = \sum_{n=0}^{M} a_n \, x^n.$$

We could also shift x to $x - a$ and get a power series centered at a. This is usually a trivial modification.

Convergence or divergence of a power series The convergence test for power series are limited in scope since they must work with a more general term involving x. The ratio test is the basic test here.

§**The Ratio test** The series Eq. (1) converges or diverges as $\rho < 1$ or $\rho > 1$, where we define

$$\begin{aligned} \rho &= \lim_{n\to\infty} \rho_n \\ \rho_n &= \frac{|a_{n+1}x^{n+1}|}{|a_n x^n|} = \frac{|a_{n+1}|}{|a_n|}|x|. \end{aligned} \tag{2}$$

Hence we can say that the power series converges or diverges as $|x| < |x_0|$ or $|x| > |x_0|$, with

$$|x_0| = |\lim_{n\to\infty} \frac{a_n}{a_{n+1}}|. \tag{3}$$

This provides a region of convergence $-x_0 < x < x_0$, this is called the interval of convergence. When we learn about complex series (where $x \to z$, with z a complex variable), the region of convergence is called the radius of convergence.

§ **Some examples**

$$P(x) = \sum_{n=0}^{\infty} \{ \frac{(-x)^n}{2^n}, \quad \frac{(-x)^n}{n}, \quad \frac{(-x)^n}{n!} \}$$

§**Comments on operations with power series** If some power series are known to be convergent, we can take many liberties with them.

- Add two power series to get another power series
- Multiply two power series to get another power series
- Integrate or differentiate the power series w.r.t. x

§**Representing a given function by a power series** This is an important concept: a given function (think $\sin(x)$ for example) can be represented by a *unique* power series within some region where the power series converges. This means there is one and only one power series for a given function.

In practice we can do a Taylor expansion to generate an infinite series.

§**Taylor and Maclaurin series** Given a function $f(x)$ satisfying the condition of being differentiable any number of times, we can expand it around a point $x = a$ as follows

$$f(x) = f(a) + (x-a)f'(a) + (x-a)^2 \frac{f''(a)}{2!} + \ldots + (x-a)^n \frac{f^{(n)}(a)}{n!} + \ldots$$

where the n^{th} derivative of f

$$f^{(n)}(a) = \left(\frac{d^n f(x)}{dx^n} \right)_{x=a}.$$

This called the Taylor series of the function around $x = a$.

If we specialize to $a = 0$ we get the Maclaurin series.

$$f(x) = f(0) + xf'(0) + x^2 \frac{f''(0)}{2!} + \ldots + x^n \frac{f^{(n)}(0)}{n!} + \ldots$$

Comment: Maclaurin versus Taylor series We note that a power series of the type

$$\sum_{n=0}^{\infty} a_n (x-A)^n,$$

can be viewed as an expansion around $x = A$, and represents the Taylor expansion of some function. If we set $A = 0$ we get a shifted series $\sum_{n=0}^{\infty} a_n x^n$, this is called the Maclaurin series.

Given a function $f(x)$ we could expand it around any point A (giving a Taylor expansion of f), if we choose $A = 0$ we get the Maclaurin series.

One simple example. Consider $f(x) = e^x$. It has a Taylor expansion around an arbitrary point a

$$f(x) = \sum_{n=0}^{\infty} (x-a)^n b_n,$$

where $b_n = e^a/n!$. (Check this.)

Now *the same function* has a Maclaurin expansion (by definition around $x = 0$)

$$f(x) = \sum_{n=0}^{\infty} x^n c_n,$$

where the coefficients $c_n = 1/n!$. Clearly $c_n \neq b_n$ for $a \neq 0$.

§Expanding functions in a Taylor series is to apply the formula

$$f(x-a) = \sum_{n=0}^{\infty} (x-a)^n \frac{f^{(n)}(a)}{n!}$$

where the nth derivative is defined as

$$f^{(n)}(a) = \left(\frac{d^n f(x)}{dx^n} \right)_{x=a}.$$

In practice we can expand out to some order M and call the rest as the remainder, thus

$$f(x-a) = \sum_{n=0}^{M} (x-a)^n \frac{f^{(n)}(a)}{n!} + R_M(a).$$

A formula to estimate the remainder R_M can be found from complex integration theory (done later). We record the answer here

$$(x-a)^{M+1} \frac{f^{(M+1)}(c)}{(M+1)!},$$

where c is a (certain) point lying in the interval of convergence.

§Remainder for alternating series:

We saw that with the definition

$$S_\infty = a_0 - a_1 + a_2 - a_3 + \ldots$$

(we could also multiply a_j by x^j if needed, but let us drop it for now), we could easily establish a bound:

$$S_{2n} \leq S_\infty \leq S_{2n} + a_{2n+2}$$

This also leads to an estimate of the remainder for an alternating series. We may rewrite this as

$$|S_\infty - S_{2n}| \leq a_{2n+2}.$$

Hence summing the series to order $2m$ comes with a remainder that is known.

§Example:

We can use this brute force formula only in a few simple cases such as the exponential series.

$$f(x) = e^x,$$

where an expansion around $x = a$ is simple, it requires $f^{(n)}(a) = e^a$ by definition. Hence we get

$$e^x = \sum_{n=0}^{\infty} (x-a)^n \frac{e^a}{n!},$$

as previously argued.

Comment on more complicated functions

It is a good idea to take a few standard power series expansions for functions that are "atoms" (i.e. modules) for complicated functions, and then to plug in. Consider the following examples.

$$\sinh(x) = \frac{1}{2}(e^x - e^{-x}) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1!)}$$

$$\cosh(x) = \frac{1}{2}(e^x + e^{-x}) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n!)}$$

$$\sin(x) = (-i)\sinh(ix),$$

$$\cos(x) = \cosh(ix)$$

Another useful operation is division or multiplication of two series to get another series. We will see such problems in the HW # 2

It is mandatory to know the series for the following simple functions

$$
\begin{aligned}
\sin(x) &= x - x^3/3! + x^5/5! - \ldots \\
\cos(x) &= 1 - x^2/2! + x^4/4! - \ldots \\
e^x &= \sum_{j=0}^{\infty} x^j/j! \\
\log(1+x) &= x - x^2/2 + x^3/3 - \ldots \\
(1+x)^n &= \sum_{j=0}^{n} {}^nC_j \; x^j \\
(1+x)^\alpha &= \sum_{j=0}^{\infty} {}^\alpha C_j \; x^j \;\; \text{if } \alpha \neq \text{integer}
\end{aligned}
\tag{4}
$$

where the binomical coefficient

$$
\begin{aligned}
{}^nC_j &= \frac{n!}{j!(n-j)!} \\
{}^\alpha C_j &= \frac{\Gamma(\alpha+1)}{j!\Gamma(\alpha+1-j)!}
\end{aligned}
\tag{5}
$$

where $\Gamma(\alpha+1)$ is the Γ function. It satisfies $\Gamma(x+1) = x\Gamma(x)$ for any x and for integer x coincides with the factorial $\Gamma(n+1) = n!$

Note that Boas uses a different notation for the combinatorial function. The two notations are summarized as

$$
{}^nC_j \leftrightarrow \begin{bmatrix} j \\ n \end{bmatrix} \tag{6}
$$

An example of the binomial coefficient for fractional index is useful to record

$$
{}^\alpha C_j = \begin{bmatrix} j \\ \alpha \end{bmatrix} = \frac{(\alpha)(\alpha-1)\ldots(\alpha-j+1)}{j(j-1)(j-2)\ldots 1} \tag{7}
$$

The case α equal to $\frac{1}{2}$ arises when we expand

$$
(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + \ldots + {}^{\frac{1}{2}}C_j \; x^j + \ldots .
$$

We can specialize Eq (7) further and get

$$
{}^{\frac{1}{2}}C_j = \begin{bmatrix} j \\ \frac{1}{2} \end{bmatrix} = \frac{(\frac{1}{2})(\frac{1}{2}-1)\ldots(\frac{1}{2}-j+1)}{j(j-1)(j-2)\ldots 1} \tag{8}
$$

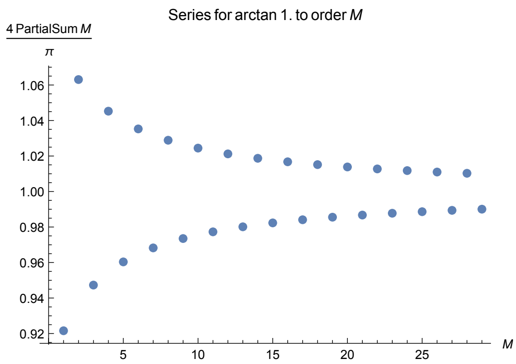

Figure 1: The partial sum $4/\pi \times S_M$ of the series for arctan(1) against the order M. We see that there are two sequences that seem to converge to the correct answer 1. The sequence with M odd lies below the

We may pull out minus signs and use the double factorial notation

$$(2j+1)!! = (2j+1)(2j-1)\dots 1,$$

and obtain

$$^{\frac{1}{2}}C_j = (-1)^{n-1}\frac{(2n-3)!!}{n!2^n}. \tag{9}$$

§An interesting method involving integration

Consider calculating the series for $\arctan(x)$. We know that

$$\arctan(x) = \int_0^x \frac{dt}{1+t^2},$$

hence we can expand the integrand and integrate term by term. Thus we get

$$\begin{aligned}\arctan(x) &= \int_0^x dt\left(1-t^2+t^4-t^6+t^8-\dots\right)\\ &= x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\frac{x^9}{9}-\dots\end{aligned} \tag{10}$$

An interesting application of this series is to calculate π from the formula $\arctan(1) = \pi/4$. We can take the series for $4/\pi \arctan(1)$ and truncate it at the Mth order for various values of $M \in (3, 31)$, and find the plot:

§**Examples of error estimates**

In the case of an alternating series we can use the theorem

$$|S_\infty - S_{2n}| \le a_{2n+2},$$

with

$$S_{2n} = a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + \ldots + a_{2n} - a_{2n+1}$$

as we proved in class. we can apply this to the series

$$1 = \frac{4}{\pi}\arctan(1) = \frac{4}{\pi}\sum_{j=1}^{\infty}(-1)^{j+1}\frac{1}{2j-1}.$$

From a computation we find $S_{16} = 1 + .018708$, and the theorem says that the partial sum $|S_{16} - S| \leq a_{18} = 0.0363783$.

The alternating series test is very useful for the trigonometric functions. For example

$$\sin(x) = x - x^3/3! + x^5/5! - \ldots,$$

so that if we want to approximate by the first two terms only, the remainder is $x^5/120$. This is small (.26666) for a fairly large value of $x \sim 2$.

An example of estimating the remainder for the non-alternating series next. Consider

$$S = \sum_{n=0}^{\infty} a_n x^n, \quad \text{assumed convergent for } |x| < 1 \tag{11}$$

and with $a_{n+1} < a_n$, then we can write

$$\begin{aligned} |S - \sum_{n=0}^{N} a_n x^n| &= |\sum_{n=N+1}^{\infty} a_n x^n| \leq a_{N+1}|\sum_{n=N+1}^{\infty} x^n|, \\ &= |a_{N+1}x^{N+1}|/(1-|x|). \end{aligned} \tag{12}$$

This can be helpful if $|x| \ll 1$ since x^N becomes small rapidly with increasing N.