Physics 116A- Winter 2018

Mathematical Methods 116 A

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Notes on two summable finite series

§We discussed two interesting series in class on the 8th January. I quoted two results without proof. I show how the results are obtained.

The series in question are

$$A(N, s) = \sum_{n=0}^{N} n^s,$$

with $s = 1, 2 \ldots$.

Let us calculate the exact series for the first two cases.

§s=1

We write

$$A(N, 1) = 1 + 2 + 3 + \ldots + (N - 1) + N.$$

Now reverse the series and rewrite

$$A(N, 1) = N + (N - 1) + \ldots + 2 + 1.$$

Add the two expressions term by term, thus

$$2A(N, 1) = (N + 1) + (N + 1) + \ldots (N + 1).$$

There are N terms on the right and hence simplifying we get

$$A(N, 1) = \frac{N(N + 1)}{2}.$$

§s=2

This is a bit more tricky and needs a double series expression. Start with

$$\begin{aligned} A(N, 2) &= \sum_{n=0}^{N} n^2 \\ &= \sum_{n=0}^{N} \sum_{k=0}^{n} n \end{aligned} \tag{1}$$

Here we used $\sum_{k=0}^{n} = n$. Let us now interchange the order of the n and k sums in Eq.(1).

$$A(N,2) = \sum_{k=0}^{N} \sum_{n=k}^{N} n$$

Now use

$$\sum_{n=k}^{N} n = A(N,1) - A(k-1,1) = \frac{1}{2}\left(N(N+1) - k(k-1)\right).$$

Substituting into Eq.(1) we get

$$\begin{aligned} A(N,2) &= \sum_{k=0}^{N} \frac{1}{2} N(N+1) + \sum_{k=0}^{N} \frac{k}{2} - \sum_{k=0}^{N} \frac{k^2}{2} \\ &= \frac{1}{2} N^2(N+1) + \frac{1}{4} N(N+1) - \frac{1}{2} A(N,2) \end{aligned}$$

Simplifying this we get

$$\begin{aligned} A(N,2) &= \frac{2}{3} N(N+1) \times \left(\frac{N}{2} + \frac{1}{4}\right) \\ &= \frac{1}{6} N(N+1)(2N+1). \end{aligned} \qquad (2)$$

This is the exact answer.
As shown in class we can get an approximate result by replacing

$$\sum_{n=0}^{N} n^s \sim \int_0^N dn\, n^s = \frac{N^{s+1}}{s+1}.$$

This agrees with the leading terms from the exact results as you can verify easily.