Physics 116A- Winter 2018

Mathematical Methods 116 A

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Notes on convergence tests for series

§: **Alternating series**

The convergence test for alternating series given in Boas can be reformulated as follows:

Let S_∞ stand for the alternating infinite series

$$S_\infty = a_0 - a_1 + a_2 - a_3 + \ldots,$$

with the terms

$$a_n > 0$$

and a monotonically decreases to zero, i.e.

$$a_{n+1} < a_n; \ \lim_{n\to\infty} a_n = 0.$$

With the above conditions, the series S_∞ converges.

Since this is an important test for convergence of the alternating series, it is worth understanding better the origin of the theorem, and also its proof (missing in Boas's book).

§**Initial Comments** This theorem is valid if the first several terms are ill behaved (i.e. do not satisfy the stated conditions), provided that for large enough index n, the a_n settle down to satisfy these conditions. This follows from the fact that we can throw out any *finite* number of terms from a series, and the convergence is unaffected.

§**Proof** Let us define a partial sum for integer m

$$S_{2m} = a_0 - a_1 + \ldots + a_{2m} - a_{2m+1}.$$

As $m \to \infty$ this is becomes the required sum $S_{2m} \to S_\infty$.

We will now consider two integers $m > n$ so that

$$S_{2m} - S_{2n} = a_{2n+2} - a_{2n+3} + \ldots + a_{2m} - a_{2m+1}.$$

This is the sum over positive terms and hence

$$S_{2m} - S_{2n} \geq 0.$$

We can safely rearrange this finite sum as

$$S_{2m}-S_{2n} = a_{2n+2}-(a_{2n+3}-a_{2n+4})-(a_{2n+5}-a_{2n+6})-\ldots-(a_{2m-1}-a_{2m})-a_{2m+1}.$$

From the monotonic decrease condition all the terms in brackets are positive $a_{2j+1} - a_{2j} > 0$ and hence we fine the bound

$$S_{2m} - S_{2n} \leq a_{2n+2} - a_{2m+1}.$$

Combining the two bounds we get

$$S_{2n} \leq S_{2m} \leq S_{2n} + (a_{2n+2} - a_{2m+1}).$$

We keep n finite and take $m \to \infty$ and hence get

$$S_{2n} \leq S_\infty \leq S_{2n} + a_{2n+2}. \quad ......(A)$$

We have used $a_{2m+1} \to 0$ from monotonicity of the $a's$. We can rearrange Eq(A) and write it in the form of a useful bound

$$0 \leq (S - S_{2n}) \leq a_{2n+2}.$$

This is slightly better bound than Eq. (14.3) in Boas (page 34). Proceeding further we may set $n = 0$ where $S_0 = a_0 - a_1$, and thus

$$a_0 - a_1 \leq S_\infty \leq a_0 - a_1 + a_2 - a_\infty. \quad ....(B)$$

This proves the finiteness of S_∞, it is a positive number bounded from below and above by positive finite numbers. (I note that Boas does not provide the bound Eq. (B) in the textbook- please check the bound in some examples and make sure our calculation is sensible.)

§**Dangers of rearranging an infinite alternating series**

An alternating series $S = \sum_n (-1)^n a_n < \infty$ can be rearranged (i.e. its terms can be shuffled around) provided it is *absolutely* convergent, i.e. $S_{absolute} = \sum_n |a_n| < \infty$. For many alternating series this is not true, e.g. $\sum_n (-1)^n/n$ is convergent but $\sum_n 1/n$ is divergent. An example where it *is true* would be $\sum_n (-1)^n/n^2$. In the above example we rearranged the terms of a finite series, which is safe because of its finiteness. However in the case of an infinite alternating series, where the absolute series does not converge, there is a dangerous theorem which says that by rearranging the terms, one can make the series tend to *any real value* one chooses. I will not discuss this in class, but interested students can look up examples in advanced books.

One example (taken from wiki) can help see the dangers. We know from the series for $\log(1+x)$ that

$$\log(2) = 1 - 1/2 + 1/3 - 1/4 + \ldots$$

By rearranging we can "apparently" (and clearly wrongly) prove that $\log(2) = 0$. To see this bravely (and wrongly, as we will show) rearrange the series and write

$$\log(2) = (1 - 1/2) - 1/4 + (1/3 - 1/6) - 1/8 + (1/5 - 1/10) - 1/12 + \ldots$$

or by simplifying the terms in brackets

$$\log(2) = 1/2 - 1/4 + 1/6 - 1/8 + 1/10 - 1/12 + \ldots$$

or taking out a common factor

$$\log(2) = \frac{1}{2}\left(1 - 1/2 + 1/3 - 1/4 + 1/5 - \ldots\right)$$

and thus fallaciously conclude that

$$\log(2) = (?!)\frac{1}{2}\log(2).$$