Physics 116A- Winter 2018

Mathematical Methods in Physics 116 A

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Lecture #1. Notes on convergence tests for series

§Introduction

- Geometric Series: Its origin in physical problems:

  (i)$1 + 2 + 2^2 + \ldots$ represents fission of single nucleus. Divergent series. Another example $h + h/2 + h/4 + \ldots$, the successive heights of a dropping ball gives a convergent series.

  (ii) Infinite series

  $$S = a(r + r^2 + r^3 + \ldots)$$

  (iii) Partial sum

  $$S_n = a(r + r^2 + \ldots + r^n)$$

  and by a simple calculation

  $$S_n = a(r^{n+1} - r)/(1 - r).$$

  Limiting process gives infinite series.

  $$\lim_{n->\infty} S_n = S.$$

  Limit exists if $r < 1$, hence series converges for $r < 1$.

- Uses of geometric series- rationalizing irrational numbers

  $$2.31429, 21429, 21429, \ldots = 2 + \frac{1}{10} + 21429\{10^{-5} + 10^{-10} + 10^{-15} + \ldots\}$$

  $$= 2 + \frac{1}{10} + 21429 \times \frac{10^{-5}}{1 - 10^{-5}} = \frac{257141}{111110}$$

- A frequently occurring finite series:

  $$A(N, s) = \sum_{j=1,N} j^s.$$

  $A(N, 1) = N(N + 1)/2$. (Prove it)

- Sequences and limits of sequences: Example $a_n = \frac{\sqrt{n^4+1}}{3n^3-2n^2+1}$. We can easily see that $\lim_{n\to\infty} a_n \to \frac{1}{3n}$. Question. Can you find the next term? i.e. $\lim_{n\to\infty} a_n \to \frac{1}{3n}\left(1 + O(\frac{1}{n})\right)$, what is this correction term?

  Examples to work out

$$a_n = n^{1/n}, \log(n)/n, (1 + a/n)^n$$

§**Comments on divergent and convergent series**

- Let us define a partial sum $S_n \equiv \sum_{j=1,n} a_j$. It is a finite sum by construction. A series converges if the sequence of partial sums $S_1, S_2, \ldots S_n \ldots$ has a finite limit as $n \to \infty$
- A common sense idea is that a series can be tested for convergence after throwing out any finite number of terms.
- Another commonsense idea is that an alternating series has a better chance of converging than a fixed sign series.

§**Different tests for convergence or divergence.**
We will write

$$S = \sum_{n=1}^{\infty} a_n.$$

- (1) Preliminary test for divergence.
  If $\lim_{n\to\infty} a_n \neq 0$ the series S diverges.

  Examples to discuss :

$$a_n = \frac{3^n}{1 + 2^n + 3^n}, \frac{n! + 3n}{(n+1)!},$$

- (2) Absolute convergence test (for series of positive terms) by comparison.

  If we know that the series

$$S_1 = \sum_{j=1}^{\infty} m_j,$$

  with $m_j > 0$ is convergent, and if a series

$$S_2 = \sum_{j=1}^{\infty} a_j,$$

has positive coefficients satisfying

$$a_j \leq m_j,$$

then S_2 is also convergent.

It is also obvious that if a_j do not have a fixed sign, but $|a_j| < m_j$, then too S_2 is convergent. (Why is this so obvious?)

Examples to discuss:

$$S_1 = \sum_{j=1}^{\infty} \frac{1}{j!},$$

This is easily summed and we know the answer is $e = 2.71828\ldots$. We can use it to show the convergence of

$$S_2 = \sum_{j=1}^{\infty} \frac{1}{(j+6) \times j!},$$

Another example: If we take $S_2 = \sum_{j=1}^{\infty} \frac{1}{j!}$, i.e. our old S_1 and we want to know whether it converges (after erasing all memory of previous calculations!!) we can try as S_1 the geometric series

$$S_1 = \sum_{j=0}^{\infty} \frac{1}{2^j},$$

which converges to 2. Now the comparison of the n^{th} term goes as

$$\frac{1}{n!} < \frac{1}{r^n},$$

which is true for $n \geq 3$. Hence S_2 converges with S_1.

- (2') Almost as a corollary we can say that if S_1 is known to be divergent, and if $|a_j| > m_j$ then $S_2 = \sum_{j=1}^{\infty} |a_j|$, is also divergent. Notice that S_2 is now forced to have only positive terms for the theorem to go through. This implies that if the absolute value of a_j is removed, the series S_2 has oscillating terms, and might converge for certain cases.

- (3) Integral test:

  A series with positive terms

$$S = \sum_{n=1}^{\infty} a(n)$$

converges or diverges with the integral

$$S_I = \int_{arb}^{\infty} a(n)\, dn$$

where arb is an arbitrary non-infinite variable and the evaluation is only done at the top limit. The integral is calculated assuming n is continuous.

Rationale See figures.

Example 1: (from quiz) Series

$$S = \sum_{n=1}^{\infty} \frac{1}{(1+3n^2+n^4)^{1/4}},$$

converges or diverges with integral

$$S_I = \int_{x_0}^{\infty} \frac{dx}{(1+3x^2+x^4)^{1/4}}$$

As we know the answer is divergent.

Example 2.

Reimann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Integral test

$$S_I = \int_{x_0}^{\infty} \frac{dx}{x^s} = \frac{1}{1-s} \left(\frac{1}{x^{s-1}} \right)_{x_0}^{\infty}$$

Converges for $s > 1$. This includes the very delicate case $s = 1+\epsilon$ with $\epsilon = .0000001$ for instance.

Exs

$$\sum_{n=1}^{\infty} \left(\frac{1}{n \log(n)}, \frac{1}{n(1+\log(n))^{1/3}}, \frac{e^n}{9+e^{2n}} \right)$$

- (4) The Ratio Test This is a pretty general test. Given a series

$$S = \sum_{n=1}^{\infty} a(n)$$

we can define $\rho_n = |\frac{a(n+1)}{a(n)}|$ and its limiting value $\rho = \lim_{n\to\infty} \rho_n$. The spirit of this test is a comparison with the geometric series.

We can say that

$$\rho < 1 (or \rho > 1)$$

implies convergence (or divergence). The case $\rho = 1$ is undecided by this test.

Examples

$$a_n = e^n/n!, 3n!/(n!2n!)$$

- (5) A special comparison test.

  Suppose $S_1 = \sum_{n=1}^{\infty} b_n$ with $b_n > 0$ is convergent, and we have $S_2 = \sum_{n=1}^{\infty} a_n$ with $a_n > 0$, with the property that $\lim_{n\to\infty} b_n/a_n \to$ constant, then S_2 is convergent. Here and below the constant is assumed non-zero.

  Example:

  $$b_n = 1/n^2; \;\; a_n = 2/\sqrt{6n^4 + 3n^3 + 2n^2 + 1}$$

- (5') A corollary of the above is that: if $S_1 = \sum_{n=1}^{\infty} b_n$ with $b_n > 0$ is divergent, and we have $S_2 = \sum_{n=1}^{\infty} a_n$ with $a_n > 0$, with the property that $\lim_{n\to\infty} b_n/a_n \to$ constant, then S_2 is divergent.

  Example:

  $$b_n = 1/n; \;\; a_n = 2/\sqrt{6n^2 + 1}$$