Physics 116A- Winter 2018

Mathematical Methods 116 A

S. Shastry, Feb 6, 2018

Notes on Vectors

§**Vectors Linear Algebra (LA) and Geometric views**

In LA, a vector can be defined after discussing a cartesian coordinate system in m dimensions. We start with a point

$$P = \{x_1, x_2, \ldots, x_m\}$$

and end up at a point

$$Q = \{x'_1, x'_2, \ldots, x'_m\}.$$

Thus P and Q are then the initial and final points of a vector which is defined as

$$\vec{a} = (x'_1 - x_1, x'_2 - x_2, \ldots x'_m - x_m).$$

Firstly note that I have a slightly different bracket around $\vec{a}$ as compared to x. This is intentional. The vector is defined as the separation between two points, and hence is unchanged if we shift x and x' by the same amount, i.e. the same vector is also equal to

$$\vec{a} = (y'_1 - y_1, y'_2 - y_2, \ldots y'_m - y_m),$$

if we translate $P \to P + \vec{b}$ and $Q \to Q + \vec{b}$ with a common vector $\vec{b}$.

This is simple but has aclear cut geometrical meaning. A vector is unchanged upon translation.

Examples

§Various properties of vectors

- 1-d, 2-d, ...md vectors. Direction and length. Picturing vectors.
- Basis $\hat{e}_j$ and $\vec{r} = \sum_j x_j \hat{e}_j$.
  We often write
  $$\vec{A} = \sum_j A_j \hat{e}_j.$$
- Length of a vector $|\vec{r}| = \sqrt{\sum_j x_j^2}$. A unit vector has only a direction.
  $$|A| = \sum_j \sqrt{A_j^2}.$$
- Multiplying vectors by a constant. Negative of a vector
- Adding vectors.: Geometrically and algebraically

§**Multiplying Vectors: dot product**

$$\hat{e}_i.\hat{e}_j = \delta_{ij}$$

$$\vec{A}.\vec{B} = |A||B|\cos(\theta)$$

§**Vectors in 2-d or 3-d: cross products and determinants**

$$\hat{e}_i \times \hat{e}_j = \epsilon_{ijk}\hat{e}_k$$

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A} = \hat{C}|A||B|\sin(\theta),$$

where $\vec{A}, \vec{B}, \hat{C}$ form a right handed triad, and $\hat{C}$ is a unit vector.

Determinant formula for cross product.

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}, \tag{1}$$

Shows vanishing of cross product for parallel vectors. To see this assume $\vec{B} = c\vec{A}$, i.e. $B_x = cA_x, B_y = cA_y, B_z = cA_z$ and hence

$$\vec{A} \times c\vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ cA_x & cA_y & cA_z \end{vmatrix} = 0, \tag{2}$$

where we can pull out a common factor of c from the third row, and use the fact from determinants, that two identical rows implies a vanishing determinant.

One more explicit way to calculate a cross product is to use the expansion in terms of the unit vectors $\hat{x}, \hat{y}, \hat{z}$ pointing in the three directions:

$$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z},$$

and a similar expansion for $\vec{B}$, and then use the familiar but important basic cross products

$$\hat{x} \times \hat{y} = \hat{z}, \quad \hat{y} \times \hat{z} = \hat{x}, \quad \hat{z} \times \hat{x} = \hat{y},$$

also $\hat{a} \times \hat{a} = 0$, and the antisymmetry of the cross product, $\hat{a} \times \hat{b} = \hat{b} \times \hat{a}$, where a one of the three directions.

§**Vectors and geometry of lines and planes**

Lines

Let us first imagine 2-dimensions

A straight line through a given point $\vec{r}_0 = \{x_0, y_0\}$ and parallel to a fixed vector $\vec{A} = \{a, b\}$ is given by

$$\vec{r} = \vec{r}_0 + \vec{A}\, t,$$

where t is a scale parameter.

To see this, take components. $x = x_0 + at$ and $y = y_0 + bt$, so that we can eliminate t and write

$$m = \frac{b}{a} = \frac{y - y_0}{x - x_0},$$

which becomes on cross multiplication the familiar equation of a line in 2-d

$$y = y_0 + m(x - x_0).$$

Three and higher dimensions gives the same formula.

§**Planes**

The equation of a plane passing through a point $\vec{r}_0 = \{x_0, y_0, z_0\}$, with a normal $\vec{A}$ is given by

$$(\vec{r} - \vec{r}_0).\vec{A} = 0.$$

This is clearly the condition for the perpendicularity of $\vec{A}$ to any vector $\vec{r} - \vec{r}_0$. Thus geometrically this implies that we have a collection of points $\vec{r}$ in three dimensions, such that $\vec{A}$ is perp to $\vec{r} - \vec{r}_0$. (Picture)

This is easy to expand out as

$$xA_x + yA_y + zA_z = d,$$

where $d = x_0 A_x + y_0 A_y + z_0 A_z$. This is the equation of a plane. Intercepts along three axes are d/A_j. (Picture)

§**Cross products and planes**

We saw geometrically that the vector $\vec{C}$ found by taking the cross product $\vec{C} = \vec{A} \times \vec{B}$ satisfies

$$\vec{C}.\vec{A} = 0, \quad \vec{C}.\vec{B} = 0$$

{ Recall our definition of the cross product:}

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A} = \hat{C}|A||B|\sin(\theta),$$

where $\vec{A}, \vec{B}, \hat{C}$ form a right handed triad, and

$$\hat{C} = \frac{1}{|C|}\vec{C}$$

is a unit vector. (Can we prove this?)

We can also define the cross product algebraically from the determinant formula which gives

$$\vec{C} = \vec{A} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}, \qquad (3)$$

Expanding out we calculate the components of $\vec{C}$

$$\vec{C} = \hat{x}(A_y B_z - A_z B_y) + \hat{y}(A_z B_x - A_x B_z) + \hat{z}(A_x B_y - A_y B_x),$$

Now calculate the dot product

$$\vec{C}.\vec{A} = A_x(A_y B_z - A_z B_y) + A_y(A_z B_x - A_x B_z) + A_z(A_x B_y - A_y B_x),$$

which vanishes on expanding out the brackets. Similarly we can show $\vec{C}.\vec{B} = 0$.

Question: Can we use this to define a set of planes given two vectors $\vec{A}$ and $\vec{B}$?

Answer:

Let us consider example problem:

Find the equation of a plane containing the vectors

$$\vec{A} = (2, 3, 1), \quad \vec{B} = (-1, 3, 7).$$

Calculate cross product

$$\vec{C} = (18, -15, 9).$$

The set of points satisfying

$$\vec{r}.\vec{C} = d,$$

generates a series of parallel planes. Explicitly

$$18x - 15y + 9z = d$$

with different values of d generates parallel planes.

Next Question: We know geometrically that a plane is defined by 3 points. Can we use vectors to find a unique plane given three points.

Note: Two points defines a vector so we been given two vectors and hence from above discussion we can find a set of planes for sure. Additionally we are given three points, any one of which can be used to determine which of the set of planes is the one we seek.

Example: Let the three points be

$$P = \{1, 0, 1\}, \;\; Q = \{-2, 4, 1\}, \;\; R = \{0, 3, 5\}$$

where the coordinates are relative to some common origin O. The we define two vectors $\vec{PQ} = Q - P = (-3, 4, 0)$ and $\vec{PR} = R - P = (-1, 3, 4)$. We can now define the normal as

$$\vec{N} = \vec{PQ} \times \vec{PR} = (16, 12, -5),$$

and hence our equation of a plane passing through point P would be

$$\vec{N}.(\vec{r} - \vec{P}) = 0,$$

where $\vec{P} = (1, 0, 1)$. (: Question:What is the difference between $\vec{P}$ and P?)

Expand out..

Question: Should we worry about passing through point Q?

Quick review: Note the very compact definitions below.

Equation of a line:

$$\vec{r} = \vec{r}_0 + t\vec{A}, \ldots (Line)$$

gives a line passing through $\vec{r}_0$ which is parallel to $\vec{A}$.

Equation of a plane

$$(\vec{r} - \vec{r}_0).\vec{A} = 0, \ldots (Plane)$$

gives a plane that passes through $\vec{r}_0$ and has a normal parallel to $\vec{A}$.

Using these we can solve some important and commonly occurring problems:

Distance between a line and a point

Question: Find the distance between a point $\vec{B}$ and the line above.

Answer: Pick any point on the line, say $\vec{r}_0$. Form the vector $\vec{C} = \vec{B} - \vec{r}_0$. Now $|C|$ is not the shortest distance, which is a normal drawn from $\vec{B}$ to the line. To get at that we now take

$$|\vec{C} \times \vec{A}|/|A| = |C| \sin(\theta),$$

this gives the required answer.

Distance between a plane and a point

Question: Find the distance between a point $\vec{B}$ and the plane above.

Answer: Take the vector joining $\vec{B}$ and any pint on the plane, say $\vec{r}_0$, thus form

$$\vec{C} = \vec{B} - \vec{r}_0.$$

This vector is not the shortest vector between the plane and the point, which would be a normal drawn from $\vec{B}$ on to the plane. The length of that can be found immediately from

$$|\vec{C}.\vec{A}|/|A|.$$

Distance between two skew lines

Find the distance between two lines:

$$\vec{r} = \vec{r}_0 + t\vec{A},$$

$$\vec{r} = \vec{r}_1 + t\vec{B}$$

where A, B are at some general angle.
Answer: Form $\vec{C} = \vec{r}_1 - \vec{r}_0$, take

$$\vec{C}.(\vec{A} \times \vec{B})/|\vec{A} \times \vec{B}|.$$