Please staple the question paper to your answer sheet.

Scores

Problem No	1	2	3	4	5	6
Max Score	20	15	20	20	10	15
Score						

Mathematical Methods of Physics

Physics 116B- Spring 2018

Final Examination solution, Total 100 Points

June 11, 2018

- Please use only the scratch paper for your answers. DO NOT write anything on the question paper except your name.
- Show details of the work and box the final results.
- Answers without detailed working will not be receive any score.
- One page of notes with formulas is allowed.
- No calculators or cell phone usage please.

1. Find the general solution of the following first-order differential equation: ... [20]

$$(1+e^x)y' + 2e^x y = (1+e^x)e^x \ .$$

§**Solution** For a linear equation in the form of $y' + P(x)y = Q(x)$, we can find a solution using the formula $y = e^{-I}\int Qe^I dx + ce^{-I}$ where $I = \int P(x)dx$. In this case we obtain

$$I = \int P(x)dx = 2\log(1+e^x)$$

where $P = 2x/(1+e^x)$. A simple way to compute this integral is to use the substitution $u = 1+e^x$. Next we compute

$$\int Qe^I dx = \int e^x(1+e^x)^2 = (1+e^x)^3/3 \ ,$$

where $Q = e^x$ and again we use the same substitution to solve. Hence, the general solution is

$$\boxed{y(x) = (1+e^x)/3 + c/(1+e^x)^2} \ .$$

2. Find the general solution to the following non-linear differential equation: ... [15]

$$(2xe^{3y} + e^x) + (3x^2e^{3y} - y^2)\frac{dy}{dx} = 0 \ .$$

§**Solution**

If we suppose there exists a function, $F(x,y) =$ constant, such that $dF = Mdy + Ndx = 0$ where $M = (2xe^{3y} + e^x)$ and $N = 3x^2e^{3y} - y^2$, we can solve the equation using the following method. First we check this equation is exactly solvable, i.e., $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial y}$:

$$\frac{\partial M}{\partial x} = 6xe^{3y} = \frac{\partial N}{\partial y} \ .$$

Since it is exactly solvable, next we integrate

$$\int Mdy = x^2e^{3y} + y^3/3 + h(x)$$

and

$$\int N dx = x^2 e^{3y} + e^x + g(y)$$

and we set integrals equal such that $h(x) = e^x$ and $g(y) = -y^3/3$. Hence, the general solution is

$$\boxed{F(x,y) = C = x^2 e^{3y} + e^x - y^3/3} \,.$$

3. Find the general solution of following inhomogenenous differential equation: . . . [15]

$$y'' - 2y' + y = 2\cos x \,.$$

§**Solution**

To find the complementary solution, we start with the guess that e^{rx} is a solution of the homogeneous differential equation when

$$(r^2 - 2r + 1)e^{rx} = 0 \;\Rightarrow\; (r-1)^2 = 0 \,.$$

Since the roots of the characteristic equation are equal the first solution is e^x and the second independent solution is xe^x . Recall that the real part of the solution, $Y_p = Ce^{ix}$, to an inhomogenous differential equation of the form $y'' + p(x)y' + q(x)y = 2e^{ix}$ is equivalent to the solution, y_p, of the inhomogeneous differential equation $y'' + p(x)y' + q(x)y = 2\cos(x)$. Inserting Y_p in differential equation, we obtain

$$(-e^{ix} - 2e^{ix} + e^{ix}) = 2e^{ix} \;\Rightarrow C = i \,,$$

and we find $Y_p = i\cos(x) - \sin(x)$ where the real part is $-\sin(x)$s. Hence, the general solution is

$$\boxed{y(x) = Ae^x + Bxe^x - \sin(x)} \,.$$

4. By using the method Laplace Transforms find the general solution of following inhomogenenous differential equation: . . . [20]

$$y'' + 2y + y = e^{-t} - 2e^{-t} \,, \quad y_0 = 1,\; y_0' = -1 \,.$$

§**Solution** First we take the Laplace transform of both sides of the differential equation:

$$p^2Y - py_0 - y_0' + 2pY - 2y_0 + Y = \frac{1}{1+p} - \frac{2}{(1+p)^2} \ .$$

Next we insert the initial conditions and group terms:

$$(p+1)^2Y - p - 1 = \frac{1}{1+p} - \frac{2}{(1+p)^2} \ .$$

Solving for Y we obtain:

$$Y = \frac{1}{(1+p)^3} - \frac{2}{(1+p)^4} + \frac{1}{(1+p)} \ .$$

Lastly, we do the inverse Laplace transformation to find the general solution:

$$y(t) = e^{-t} + (1/2)t^2e^{-t} - (1/3)t^3e^{-t}$$

5. Find J_n for $n = 1, 2, 3$ by integrating around a contour consisting of a circle of radius 2 encircling the orgin: ... [15]

$$J_n = \frac{1}{2\pi i}\oint_{C_2} \frac{(1-z)^2}{z^n} dz \ .$$

§**Solution**

We can find the solution using residue theorem, i.e., $\int f(z)dz = 2\pi i \sum R(z)$:

$$J_1 = \frac{1}{2\pi i}\oint \frac{(1-z)^2}{z} = \frac{1}{2\pi i}\left(\oint dz/z - 2\oint dz + \oint zdz\right) = 1 \ .$$

Note that only the first contour integral is non-zero, that is $\oint dz/z = 2\pi i$.

$$J_2 = \frac{1}{2\pi i}\oint \frac{(1-z)^2}{z^2} = \frac{1}{2\pi i}\left(\oint dz/z^2 - 2\oint dz/z + \oint dz\right) = -2 \ .$$

Here only the second contour integral is non-zero.

$$J_3 = \frac{1}{2\pi i}\oint \frac{(1-z)^2}{z^2} = \frac{1}{2\pi i}\left(\oint dz/z^3 - 2\oint dz/z^2 + \oint dz/z \right) = 1 \; .$$

In this last one, we see that only that last contour integral is non-zero.

6. Find the Laurent series and the residue for the following function at the indicated point: ... [15]

$$\frac{\sin z}{z^4}, \text{ at } z = 0 \; .$$

In order to find the Laurent series, we first identify the singularities in the function. In this case there is only one singularity at $z = 0$. We note that $\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = 0$ and it is analytic for all $|z| < \infty$, so the Laurent series is

$$\frac{\sin z}{z^4} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n-3}}{(2n+1)!} \; ,$$

and therefore the residue is

$$R(0) = -1/3! \; .$$

Table of Laplace Transforms

	$y = f(t),\ t > 0$ $[y = f(t) = 0,\ t < 0]$	$Y = L(y) = F(p) = \int_0^\infty e^{-pt} f(t)\, dt$	
$L1$	1	$\dfrac{1}{p}$	$\text{Re } p > 0$
$L2$	e^{-at}	$\dfrac{1}{p+a}$	$\text{Re } (p+a) > 0$
$L3$	$\sin at$	$\dfrac{a}{p^2+a^2}$	$\text{Re } p > \lvert \text{Im } a \rvert$
$L4$	$\cos at$	$\dfrac{p}{p^2+a^2}$	$\text{Re } p > \lvert \text{Im } a \rvert$
$L5$	$t^k,\ k > -1$	$\dfrac{k!}{p^{k+1}}$ or $\dfrac{\Gamma(k+1)}{p^{k+1}}$	$\text{Re } p > 0$
$L6$	$t^k e^{-at},\ k > -1$	$\dfrac{k!}{(p+a)^{k+1}}$ or $\dfrac{\Gamma(k+1)}{(p+a)^{k+1}}$	$\text{Re } (p+a) > 0$
$L7$	$\dfrac{e^{-at} - e^{-bt}}{b-a}$	$\dfrac{1}{(p+a)(p+b)}$	$\text{Re } (p+a) > 0$ $\text{Re } (p+b) > 0$
$L8$	$\dfrac{ae^{-at} - be^{-bt}}{a-b}$	$\dfrac{p}{(p+a)(p+b)}$	$\text{Re } (p+a) > 0$ $\text{Re } (p+b) > 0$
$L9$	$\sinh at$	$\dfrac{a}{p^2-a^2}$	$\text{Re } p > \lvert \text{Re } a \rvert$
$L10$	$\cosh at$	$\dfrac{p}{p^2-a^2}$	$\text{Re } p > \lvert \text{Re } a \rvert$
$L11$	$t \sin at$	$\dfrac{2ap}{(p^2+a^2)^2}$	$\text{Re } p > \lvert \text{Im } a \rvert$
$L12$	$t \cos at$	$\dfrac{p^2-a^2}{(p^2+a^2)^2}$	$\text{Re } p > \lvert \text{Im } a \rvert$
$L13$	$e^{-at} \sin bt$	$\dfrac{b}{(p+a)^2+b^2}$	$\text{Re } (p+a) > \lvert \text{Im } b \rvert$
$L14$	$e^{-at} \cos bt$	$\dfrac{p+a}{(p+a)^2+b^2}$	$\text{Re } (p+a) > \lvert \text{Im } b \rvert$
$L15$	$1 - \cos at$	$\dfrac{a^2}{p(p^2+a^2)}$	$\text{Re } p > \lvert \text{Im } a \rvert$
$L16$	$at - \sin at$	$\dfrac{a^3}{p^2(p^2+a^2)}$	$\text{Re } p > \lvert \text{Im } a \rvert$
$L17$	$\sin at - at \cos at$	$\dfrac{2a^3}{(p^2+a^2)^2}$	$\text{Re } p > \lvert \text{Im } a \rvert$