Mathematical Methods of Physics

Physics 116B- Spring 2018

Midterm Examination,   Total 100 Points
May 3, 2018

Your Name (Capitals) ....

- Please use only the scratch paper for your answers. DO NOT write anything on the question paper except your name.
- Show details of the work and box the final results.
- Answers without detailed working will not be receive any score.
- One page of notes with formulas is allowed.
- No calculators or cell phone usage please.

1. Find the four independent *real* solutions of the linear differential equation

$$(D+1)^2(D^2+4)y = 0.$$

The result should be expressed in terms of 4 real parameters. . . . [25]

Solution: The auxiliary polynomial is factorized as $(d+1)^2(d+2i)(d-2i)$. *We have a double root at* $d = 1$ *and single roots everywhere else. We can write the answer by inspection*

$$y = A\sin(2t+\phi) + (B+Ct)e^{-t}$$

2. Given four 3-dimensional vectors $\vec{X}, \vec{Y}, \vec{P}, \vec{Q}$ calculate

$$\epsilon_{ijk}\epsilon_{imn}X_jY_kP_mQ_n,$$

where ϵ_{ijk} is the Levi-Civita tensor. . . . [10]

Solution: This is a simple application of the formula $\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$. *Applying this we get the result*

$$(\vec{X}.\vec{P})(\vec{Y}.\vec{Q}) - (\vec{X}.\vec{Q})(\vec{Y}.\vec{P})$$

3. The position vector of a moving particle moving on the surface of a cylinder, is given in terms of the three unit cartesian vectors as

$$d\vec{s} = \hat{i}dx + \hat{j}dy + \hat{k}dz.$$

We make a change of variables to cylindrical coordinates $x = r\cos(\theta), y = r\sin(\theta), z$, and note that constraint implies $\dot{r} = 0$. Using this constraint find the velocity

$$\vec{v} = \frac{d\vec{s}}{dt}$$

and acceleration

$$\vec{a} = \frac{d^2\vec{s}}{dt^2}$$

in terms of the vectors $\hat{e}_\theta$, $\hat{e_r}$, $\hat{z}$, and the derivatives of θ, z.

In the special case where the rates of change $\dot{\theta}$ and $\dot{z}$ are constant, show that the particle acceleration is non-zero and centripetal (i.e. along the radial direction). ...[35]

Solution: We can write

$$d\vec{s} = \hat{e}_r dr + r\hat{e}_\theta d\theta + \hat{z}dz,$$

and hence dividing by dt

$$\vec{v} = \hat{e}_r \dot{r} + r\hat{e}_\theta \dot{\theta} + \hat{z}\dot{z}.$$

Using the constraint $\dot{r} = 0$ we can drop the first term. To take one more derivative, we need

$$\frac{d\hat{e}_r}{dt} = \hat{e}_\theta \frac{d\theta}{dt}, \quad \frac{d\hat{e}_\theta}{dt} = \hat{e}_r \frac{d\theta}{dt}.$$

With this we obtain

$$\vec{a} = \frac{d^2\vec{s}}{dt^2} = r\left(\hat{e}_\theta \ddot{\theta} + \hat{e}_r (\dot{\theta})^2\right) + \hat{z}\ddot{z}.$$

When $\ddot{\theta} = 0$ and $\ddot{z} = 0$, we can drop the first and last terms and the remaining term is clearly radial. This is the famous centripetal force of rotation in physics- a force that comes about due to the constraint of living on a surface and not due to any other explicit force acting on the particle.

4. A dying population of insects, evolves in time with the rate of decay given by

$$\frac{dN}{dt} = -(N + cN^2)/\tau.$$

Find the population as a function of time t and $N(0)$, the initial population.

Assuming $N(0)c = 1$, show that the population at $t = \tau$ is $\frac{1}{2e-1}$ of the initial population.

...[30]

Solution: We solve the equation by separation of variables

$$\frac{dN}{N + cN^2} = -\frac{dt}{\tau}$$

so

$$\log\left(\frac{Nc}{N+c}\right) = -\frac{t}{\tau} + const,$$

so the solution is

$$N(t) = \frac{N(0)}{(1 + cN(0))e^{t/\tau} - cN(0)}.$$

We see that at $t = \tau$ with $cN(0) = 1$, the solution gives $N(\tau)/N(0) = \frac{1}{2e-1}$, as required.