PHYS 116C
Practice Midterm

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1 Problem One

Consider a object orbiting around the earth at a constant altitude and constant latitude, i.e. orbiting at a constant radius R from the earth's center and a constant angle from the north pole θ_0. Show that the velocity and acceleration are given by

$$\frac{d\boldsymbol{s}}{dt} = R\sin\theta_0\dot{\phi}\boldsymbol{e}_\phi \tag{1.1}$$

$$\frac{d^2\boldsymbol{s}}{dt^2} = R\sin\theta_0\left(-\sin\theta_0\dot{\phi}^2\boldsymbol{e}_\rho - \cos\theta_0\dot{\phi}^2\boldsymbol{e}_\theta + \ddot{\phi}\boldsymbol{e}_\phi\right) \tag{1.2}$$

Here we are using

$$x = \rho\sin\theta\cos\phi \tag{1.3}$$

$$y = \rho\sin\theta\sin\phi \tag{1.4}$$

$$z = \rho\cos\theta \tag{1.5}$$

2 Problem Two

A helicopter hovering over a target releases a payload of mass m from rest. Including linear air-drag, the differential equation describing the motions of the payload in the vertical direction is

$$m\frac{d^2y}{dt^2} = -b\frac{dy}{dt} - mg \tag{2.1}$$

Solve for the position as a function of time by first solving for the velocity as a function of time. After computing $v(t) = \dfrac{dy}{dt}$, integrate the velocity to compute the position $y(t)$. Using $v(t)$, determine the terminal velocity of the payload, i.e. the limit of the velocity as $t \to \infty$.

3 Problem Three

A very useful identity in quantum field theory and group theory is the Jacobi identity. A manifestation of the Jacobi identity in terms of Levi-Civita symbols is as follows:

$$\epsilon_{ade}\epsilon_{bcd} + \epsilon_{bde}\epsilon_{cad} + \epsilon_{cde}\epsilon_{abd} = 0 \tag{3.1}$$

Prove that this identity holds.

4 Problem Four

Prove the following identities:

$$\nabla(\boldsymbol{A} \cdot \boldsymbol{B}) = (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} + (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} + \boldsymbol{A} \times (\nabla \times \boldsymbol{B}) + \boldsymbol{B} \times (\nabla \times \boldsymbol{A}) \tag{4.1}$$

$$\nabla \times (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{A}(\nabla \cdot \boldsymbol{B}) - \boldsymbol{B}(\nabla \cdot \boldsymbol{A}) + (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} - (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} \tag{4.2}$$

[**Hint:** Recall that $\epsilon_{iab}\epsilon_{icd} = \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bd}$]

5 Problem Five

Consider a short lived, but prolific-breeding species inside a box. Suppose the rate of breeding is proportional to the square of the density of beings inside the box and that the species dies off at a rate of γ. The differential equation for the number of beings as a function of time can be written as

$$\frac{dN}{dt} = AN^2 - \gamma N \tag{5.1}$$

Find $N(t)$ given $N(0) = 1$. For what ratio of A/γ does the species remain constant, i.e $N(t) = 1$? (In case applying such an equation to a small number $N = 1$ bothers you, one can imagine that N is being measured in some large units (e.g. $N = 1$ really means a a million.)

6 Problem Six

Solve the following differential equations

$$(D^2 + 1)(D^2 - 1)y = 0 \tag{6.1}$$

$$(D^3 + D^2 - 6D)y = 0 \tag{6.2}$$

where $D = d/dx$.