PHYS 116C
Practice Midterm
Solutions

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1 Problem One

Consider a object orbiting around the earth at a constant altitude and constant latitude, i.e. orbiting at a constant radius R from the earth's center and a constant angle from the north pole θ_0. Show that the velocity and acceleration are given by

$$\frac{d\boldsymbol{s}}{dt} = R\sin\theta_0\dot{\phi}\boldsymbol{e}_\phi \tag{1.1}$$

$$\frac{d^2\boldsymbol{s}}{dt^2} = R\sin\theta_0\left(-\sin\theta_0\dot{\phi}^2\boldsymbol{e}_\rho - \cos\theta_0\dot{\phi}^2\boldsymbol{e}_\theta + \ddot{\phi}\boldsymbol{e}_\phi\right) \tag{1.2}$$

Here we are using

$$x = \rho\sin\theta\cos\phi \tag{1.3}$$

$$y = \rho\sin\theta\sin\phi \tag{1.4}$$

$$z = \rho\cos\theta \tag{1.5}$$

1.1 Solution One

First, let's compute $\boldsymbol{e}_\rho, \boldsymbol{e}_\theta, \boldsymbol{e}_\phi$. Given the relationships in Eq. (1.3) - Eq. (1.5), we find that

$$dx = (\sin\theta\cos\phi)\,d\rho + (\rho\cos\theta\cos\phi)\,d\theta + (-\rho\sin\theta\sin\phi)\,d\phi \tag{1.6}$$

$$dy = (\sin\theta\sin\phi)\,d\rho + (\rho\cos\theta\sin\phi)\,d\theta + (\rho\sin\theta\cos\phi)\,d\phi \tag{1.7}$$

$$dz = (\cos\theta)\,d\rho + (-\rho\sin\theta)\,d\theta + (0)\,d\phi \tag{1.8}$$

Given that

$$d\boldsymbol{s} = \hat{x}dx + \hat{y}dy + \hat{z}dz = \boldsymbol{e}_\rho d\rho + \rho \boldsymbol{e}_\theta d\theta + \rho \sin\theta \boldsymbol{e}_\phi d\phi \tag{1.9}$$

We can see that

$$\begin{aligned} d\boldsymbol{s} = \hat{x}\left(\sin\theta\cos\phi d\rho + \rho\cos\theta\cos\phi d\theta - \rho\sin\theta\sin\phi d\phi\right) && (1.10)\\ + \hat{y}\left(\sin\theta\sin\phi d\rho + \rho\cos\theta\sin\phi d\theta + \rho\sin\theta\cos\phi d\phi\right) && (1.11)\\ + \hat{z}\left(\cos\theta d\rho + -\rho\sin\theta d\theta\right) && (1.12)\end{aligned}$$

Hence, the unit vectors in spherical coordinates are

$$\begin{aligned} \boldsymbol{e}_\rho &= \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta && (1.13)\\ \boldsymbol{e}_\theta &= \hat{x}\cos\theta\cos\phi + \hat{y}\cos\theta\sin\phi - \hat{z}\sin\theta && (1.14)\\ \boldsymbol{e}_\phi &= -\hat{x}\sin\phi + \hat{y}\cos\phi && (1.15)\end{aligned}$$

Next, we can compute the velocity by dividing $d\boldsymbol{s}$ by dt:

$$\frac{d\boldsymbol{s}}{dt} = \boldsymbol{e}_\rho\dot{\rho} + \rho\boldsymbol{e}_\theta\dot{\theta} + \rho\sin\theta\boldsymbol{e}_\phi\dot{\phi} \tag{1.16}$$

Since $\rho = R =$ constant and $\theta = \theta_0 =$ constant, we find that

$$\frac{d\boldsymbol{s}}{dt} = R\sin\theta_0\boldsymbol{e}_\phi\dot{\phi} \tag{1.17}$$

Next, to compute the acceleration, we need to take the derivative of $d\boldsymbol{s}/dt$. In order to do so, we need to compute $d\boldsymbol{e}_\phi/dt$. This is

$$\frac{d\boldsymbol{e}_\phi}{dt} = \frac{d}{dt}\left(-\hat{x}\sin\phi + \hat{y}\cos\phi\right) = -\left(\hat{x}\cos\phi + \hat{y}\sin\phi\right)\dot{\phi} \tag{1.18}$$

Let's try to write this in terms of $\boldsymbol{e}_\rho$ and $\boldsymbol{e}_\theta$. It isn't to hard to see that

$$\sin\theta\boldsymbol{e}_\rho + \cos\theta\boldsymbol{e}_\theta = \cos\phi\hat{x} + \sin\phi\hat{y} \tag{1.19}$$

Thus, we can see that

$$\frac{d\boldsymbol{e}_\phi}{dt} = -\left(\sin\theta\boldsymbol{e}_\rho + \cos\theta\boldsymbol{e}_\theta\right)\dot{\phi} \tag{1.20}$$

Now, the acceleration is

$$\frac{d^2\boldsymbol{s}}{dt^2} = R\sin\theta_0\left(\dot{\phi}\frac{d\boldsymbol{e}_\phi}{dt} + \ddot{\phi}\boldsymbol{e}_\phi\right) = R\sin\theta_0\left(-\dot{\phi}^2\sin\theta_0\boldsymbol{e}_\rho - \dot{\phi}^2\cos\theta_0\boldsymbol{e}_\theta + \ddot{\phi}\boldsymbol{e}_\phi\right) \tag{1.21}$$

2 Problem Two

A helicopter hovering over a target releases a payload of mass m from rest. Including linear air-drag, the differential equation describing the motions of the payload in the vertical direction is

$$m\frac{d^2y}{dt^2} = -b\frac{dy}{dt} - mg \tag{2.1}$$

Solve for the position as a function of time by first solving for the velocity as a function of time. After computing $v(t) = \frac{dy}{dt}$, integrate the velocity to compute the position $y(t)$. Using $v(t)$, determine the terminal velocity of the payload, i.e. the limit of the velocity as $t \to \infty$.

2.1 Solution Two

The differential equation for the velocity is given by

$$\frac{dv}{dt} + \frac{b}{m}v = -g \tag{2.2}$$

This is solved by use of an integrating factor equal to $I(t) = e^{bt/m}$. Multiplying by $I(t)$ our differential equation can be written as

$$\frac{d}{dt}\left(e^{bt/m}v(t)\right) = -ge^{bt/m} \tag{2.3}$$

Integrating both sides, and solving for v we find

$$v(t) = -\frac{gm}{b} + \mathcal{C}e^{-bt/m} \tag{2.4}$$

Given that $v(0) = 0$, we find that $\mathcal{C} = gm/b$. Hence,

$$v(t) = \frac{gm}{b}\left(e^{-bt/m} - 1\right) \tag{2.5}$$

Next, let's integrate this to obtain $y(t)$:

$$y(t) = \frac{gm}{b}\left(-\frac{m}{b}e^{-bt/m} - t\right) + \mathcal{D} \tag{2.6}$$

Setting $y(0) = y_0$, we find that

$$\mathcal{D} = y_0 + \frac{gm^2}{b^2} \tag{2.7}$$

Therefore,

$$y(t) = y_0 + \frac{gm^2}{b^2}\left(1 - \frac{b}{m}t - e^{-bt/m}\right) \tag{2.8}$$

Lastly, we can take the limit as $t \to \infty$ of $v(t)$ to find the terminal velocity:

$$\lim_{t\to\infty} v(t) = -\frac{gm}{b} \tag{2.9}$$

Notice that this agrees with setting $dv/dt = 0$ in the differential equation for $v(t)$.

3 Problem Three

A very useful identity in quantum field theory and group theory is the Jacobi identity. A manifestation of the Jacobi identity in terms of Levi-Civita symbols is as follows:

$$\epsilon_{ade}\epsilon_{bcd} + \epsilon_{bde}\epsilon_{cad} + \epsilon_{cde}\epsilon_{abd} = 0 \tag{3.1}$$

Prove that this identity holds.

3.1 Solution Three

To prove this identity, we use

$$\epsilon_{iab}\epsilon_{icd} = \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc} \tag{3.2}$$

Before we start using this identity, let's get all repeated indices into the first position by using the anti-symmetry property of the levi-civita symbol (i.e. $\epsilon_{\ldots i \ldots j \ldots} = -\epsilon_{\ldots j \ldots i \ldots}$):

$$\epsilon_{ade}\epsilon_{bcd} + \epsilon_{bde}\epsilon_{cad} + \epsilon_{cde}\epsilon_{abd} = -\epsilon_{dae}\epsilon_{dbc} - \epsilon_{dbe}\epsilon_{dca} - \epsilon_{dce}\epsilon_{dab} \tag{3.3}$$

Next, let's use the above identity everywhere

$$\begin{aligned} \epsilon_{ade}\epsilon_{bcd} + \epsilon_{bde}\epsilon_{cad} + \epsilon_{cde}\epsilon_{abd} &= -(\delta_{ab}\delta_{ec} - \delta_{ac}\delta_{ed}) - (\delta_{dc}\delta_{ea} - \delta_{ba}\delta_{ec}) - (\delta_{ca}\delta_{eb} - \delta_{cb}\delta_{ea}) && (3.4) \\ &= -\cancel{\delta_{ab}\delta_{ec}} + \cancel{\delta_{ac}\delta_{ed}} - \cancel{\delta_{dc}\delta_{ea}} + \cancel{\delta_{ba}\delta_{ec}} - \cancel{\delta_{ca}\delta_{eb}} + \cancel{\delta_{cb}\delta_{ea}} && (3.5) \\ &= 0 && (3.6) \end{aligned}$$

4 Problem Four

Prove the following identities:

$$\nabla(\boldsymbol{A} \cdot \boldsymbol{B}) = (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} + (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} + \boldsymbol{A} \times (\nabla \times \boldsymbol{B}) + \boldsymbol{B} \times (\nabla \times \boldsymbol{A}) \tag{4.1}$$

$$\nabla \times (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{A}(\nabla \cdot \boldsymbol{B}) - \boldsymbol{B}(\nabla \cdot \boldsymbol{A}) + (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} - (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} \tag{4.2}$$

[**Hint:** Recall that $\epsilon_{iab}\epsilon_{icd} = \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}$]

4.1 Solution Four

Let's work with the first identity first. In tensor notation, this can be written as

$$\nabla\left(\boldsymbol{A} \cdot \boldsymbol{B}\right)_a = B_b \partial_a A_b + A_b \partial_a B_b = \delta_{ac}\delta_{bd}\left(B_b \partial_c A_d + A_b \partial_c B_d\right) \tag{4.3}$$

Recall that $\epsilon_{iab}\epsilon_{icd} = \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}$. We can use this identity to write

$$\delta_{ac}\delta_{bd} = \delta_{ad}\delta_{bc} + \epsilon_{iab}\epsilon_{icd} \tag{4.4}$$

Then, we find that

$$\begin{aligned}
\nabla\left(\boldsymbol{A} \cdot \boldsymbol{B}\right) &= \left(\delta_{ad}\delta_{bc} + \epsilon_{iab}\epsilon_{icd}\right)\left(B_b \partial_c A_d + A_b \partial_c B_d\right) && (4.5)\\
&= B_b \partial_b A_a + A_b \partial_b B_a + B_b \partial_c A_d \epsilon_{iab}\epsilon_{icd} + A_b \partial_c B_d \epsilon_{iab}\epsilon_{icd} && (4.6)\\
&= \left(\boldsymbol{B} \cdot \nabla\right)\boldsymbol{A} + \left(\boldsymbol{A} \cdot \nabla\right)\boldsymbol{B} + B_b\left(\nabla \times \boldsymbol{A}\right)_i \epsilon_{iab} + A_b\left(\nabla \times \boldsymbol{B}\right)_i \epsilon_{iab} && (4.7)\\
&= \left(\boldsymbol{B} \cdot \nabla\right)\boldsymbol{A} + \left(\boldsymbol{A} \cdot \nabla\right)\boldsymbol{B} + \boldsymbol{B} \times \left(\nabla \times \boldsymbol{A}\right) + \boldsymbol{A} \times \left(\nabla \times \boldsymbol{B}\right) && (4.8)
\end{aligned}$$

Which is what we set out to prove. Next, let's look at the second identity:

$$\nabla \times \left(\boldsymbol{A} \times \boldsymbol{B}\right) = \epsilon_{iab}\partial_a\left(\boldsymbol{A} \times \boldsymbol{B}\right)_b = \epsilon_{iab}\epsilon_{bcd}\partial_a\left(A_c B_d\right) = \epsilon_{iab}\epsilon_{bcd}\left(A_c \partial_a B_d + B_d \partial_a A_c\right) \tag{4.9}$$

Using the usual levi-civita identity, we find

$$\begin{aligned}
\nabla \times \left(\boldsymbol{A} \times \boldsymbol{B}\right) &= \left(\delta_{ic}\delta_{ad} - \delta_{id}\delta_{ac}\right)\left(A_c \partial_a B_d + B_d \partial_a A_c\right) && (4.10)\\
&= \boldsymbol{A}\left(\nabla \cdot \boldsymbol{B}\right) + \left(\boldsymbol{B} \cdot \nabla\right)\boldsymbol{A} - \left(\boldsymbol{A} \cdot \nabla\right)\boldsymbol{B} - \boldsymbol{B}\left(\nabla \cdot \boldsymbol{A}\right) && (4.11)
\end{aligned}$$

which is what we set out to prove.

5 Problem Five

Consider a short lived, but prolific-breeding species inside a box. Suppose the rate of breeding is proportional to the square of the density of beings inside the box and that the species dies off at a rate of γ. The differential equation for the number of beings as a function of time can be written as

$$\frac{dN}{dt} = AN^2 - \gamma N \tag{5.1}$$

Find $N(t)$ given $N(0) = 1$. For what ratio of A/γ does the species remain constant, i.e $N(t) = 1$?

5.1 Solution Five

We notice that this differential equation is separable. It separates to

$$\frac{dN}{N(AN-\gamma)} = dt \tag{5.2}$$

To integrate the left-hand-side, we use partial fractions:

$$\frac{1}{N(AN-\gamma)} = \frac{\alpha}{N} + \frac{\beta}{AN-\gamma} = \frac{N(A\alpha+\beta) - \alpha\gamma}{N(AN-\gamma)} \tag{5.3}$$

We can see that $\alpha = -1/\gamma$ and hence $\beta = -A\alpha = A/\gamma$. Therefore,

$$\int \frac{dN}{N(AN-\gamma)} = \frac{1}{\gamma}\int dN \left(-\frac{1}{N} + \frac{A}{AN-\gamma}\right) = \frac{1}{\gamma}\log\left(\frac{AN-\gamma}{N}\right) \tag{5.4}$$

We thus have

$$\frac{1}{\gamma}\log\left(\frac{AN-\gamma}{N}\right) = t + \mathcal{C} \tag{5.5}$$

Exponentiating both sides, we find

$$\frac{AN-\gamma}{N} = \mathcal{C}e^{\gamma t} \tag{5.6}$$

Solving for N we find

$$N(t) = \frac{\gamma}{A - \mathcal{C}e^{\gamma t}} \tag{5.7}$$

Forcing $N(0) = 1$, we find $\mathcal{C} = A - \gamma$. Therefore, our solution is

$$N(t) = \frac{\gamma}{\gamma e^{\gamma t} + A(1-e^{\gamma t})} = \frac{\gamma}{(\gamma - A)e^{\gamma t} + A} \tag{5.8}$$

We can see that for $\gamma = A$ that $N(t) = 1$.

6 Problem Six

Solve the following differential equations

$$(D^2 + 1)(D^2 - 1)y = 0 \tag{6.1}$$

$$(D^3 + D^2 - 6D)y = 0 \tag{6.2}$$

where $D = d/dx$.

6.1 Solution Six

To solve these problems, we guess a solution of the form

$$y(x) = Ae^{\omega x} \tag{6.3}$$

Plugging this into the first equation, we find

$$Ae^{\omega x}(\omega^2 + 1)(\omega^2 - 1) = 0 \tag{6.4}$$

For $A \neq 0$, we require that $\omega = \pm i$ or $\omega = \pm 1$. Therefore, our solution is

$$y(x) = c_1 e^{ix} + c_2 e^{-ix} + c_3 e^{x} + c_4 e^{-x} \tag{6.5}$$

We can write this in terms of real solutions as

$$y(x) = c_1' \sin(x) + c_2' \cos(x) + c_3 e^{x} + c_4 e^{-x} \tag{6.6}$$

To solve the second equation, we again guess $y = Ae^{\omega x}$. Then we find

$$Ae^{\omega x}\omega(\omega^2 + \omega - 6) = Ae^{\omega x}\omega(\omega + 3)(\omega - 2) = 0 \tag{6.7}$$

Thus, either $\omega = 0, -3, 2$. Therefore, our solution is

$$y(x) = c_1 + c_2 e^{-3x} + c_3 e^{2x} \tag{6.8}$$