Mathematical Methods of Physics 116B- Spring 2018

Physics 116B

Home Work # 6 Solutions
Posted on May 17, 2018
Due in Class Mar 24, 2018

§**Required Problems: Each problem has 10 points**

E.g. MB 19.16 means problem #16 on page 19 in the book by M. Boas, 3rd Edition.

1. MB 439.3 Using L2, or L3, and L4, verify L9 and L10.

   §**Solution**

   L9 and L10 are

$$L[\sinh(at)] = \int_0^\infty e^{-pt}\sinh(at) = \frac{a}{p^2-a^2}\ , \tag{1}$$

   and

$$L[\cosh(at)] = \int_0^\infty e^{-pt}\cosh(at) = \frac{p}{p^2-a^2}\ . \tag{2}$$

   Note that

$$\sinh(at) = \frac{e^{at}-e^{-at}}{2}\ ,$$

   so we plug this into Eq. 1 to get

$$\frac{1}{2}\int_0^\infty e^{-pt}e^{at} - \frac{1}{2}\int_0^\infty e^{-pt}e^{-at} = \frac{L[e^{at}]}{2} - \frac{L[e^{-at}]}{2} \tag{3}$$

   L2 gives Laplace transform for the first and second term on the right hand side of Eq. 3:

$$\frac{1}{2}\frac{1}{p-2} - \frac{1}{2}\frac{a}{p+a} = \boxed{\frac{a}{p^2-a^2}}\ . \tag{4}$$

   Similary we can plug $\cosh(at) = (e^{at}+e^{-at})/2$ into Eq. 2 to get

$$\frac{1}{2}\int_0^\infty e^{-pt}e^{at} + \frac{1}{2}\int_0^\infty e^{-pt}e^{-at} = \frac{L[e^{at}]}{2} + \frac{L[e^{-at}]}{2}\ , \tag{5}$$

   and using L2 we find

$$\frac{1}{2}\frac{1}{p-2} + \frac{1}{2}\frac{a}{p+a} = \boxed{\frac{p}{p^2-a^2}}\ . \tag{6}$$

2. MB 439.9 Find the inverse transform of the following:

$$\frac{5-2p}{p^2+p-2} \quad \text{Hint: Use L7 and L8.} \tag{7}$$

§**Solution**

The first step is to factor the denomator and expand numerator of Eq. 7 into two fractions:

$$\frac{5}{(p+2)(p-1)} - \frac{2p}{(p+2)(p-1)} \ . \tag{8}$$

In the above equation the inverse transformtion is given by L7 and L8 for first and second term respectively:

$$5\frac{e^{-at}-e^{-bt}}{b-a} - 2\frac{ae^{-at}-be^{-bt}}{a-b} \ , \tag{9}$$

and this simplies to

$$\boxed{e^{t}-3e^{-2t}} \ . \tag{10}$$

3. MB 439.23 Use the results which you obtain in MB 439.21 and 439.22 to find the inverse transform of $(p^2+2p-1)/(p^2+4p+5)^2$.

§**Solution**

In problems MB 439.21 and MB 439.22 we find

$$L[e^{-at}t\sin(at)] = 2b\frac{p+a}{((p+a)^2+b^2)^2} \ , \tag{11}$$

and

$$L[e^{-at}t\cos(at)] = \frac{(p+a)^2+b^2}{((p+a)^2+b^2)^2} \ , \tag{12}$$

respectively.

We can start by rewriting the numerator and denomator into square factors and expand it as follows:

$$\frac{(p+2)^2-2p-5}{((p+2)^2+1)^2} = \frac{(p+2)^2-1}{((p+2)^2+1)^2} + \frac{-2(p+2)}{((p+2)^2+1)^2} \ . \tag{13}$$

Now we use Eq. 11 and Eq. 12 to inverse Laplace transform the first term and second term on the RHS of Eq. 13 respectively:

$$\boxed{-e^{-2t}t\sin(2t)+e^{-2t}t\cos(t)} \tag{14}$$

4. MB 443.7 By using Laplace transforms, solve the following differential equation with given initial conditions:

$$y'' - 4y' + 4y = 4\ , \quad y_0 = 0\ , \quad y_0' = -2 \tag{15}$$

§**Solution**

We take the Laplace transformation of each term in Eq. 15

$$p^2Y - py_0 - y_0' - 4pY + y_0 + 4Y = \frac{1}{p} \tag{16}$$

Now plug in the value of the initial conditions and group terms:

$$(p^2 - 4p + 4)Y + 2 = \frac{1}{p}\ . \tag{17}$$

Next we factor the first term of Eq. 17 we solve for Y:

$$Y = \frac{1}{p(p-2)^2} - \frac{2}{(p-2)^2}\ . \tag{18}$$

Using L15 and L3 to inverse transform the first and second respectively term on the RHS of Eq 18:

$$\boxed{1 - e^{2t}} \tag{19}$$

where in L15 and L3 $a = 2i$.

5. MB 443.18 By using Laplace transforms, solve the following differential equations with given initial conditions:

$$y'' - 4y = 3\ , \quad y_0 = 1\ , \quad y_0' = -3 \tag{20}$$

§**Solution**

We take the Laplace transformation of each term in Eq. 20

$$p^2Y - py_0 - y_0' - 4Y = \frac{1}{p+1} \tag{21}$$

Now plug in the value of the initial conditions and group terms:

$$(p^2 - 4)Y - p - 3 = \frac{1}{p+1}\ . \tag{22}$$

Next we solve for Y:

$$Y = \frac{1}{(p+1)(p^2-4)} + \frac{p+3}{(p^2-4)}\ . \tag{23}$$

Now we use partial fraction decompostion on the first term on the RHS the Eq. 29:

$$Y = \frac{1/4}{p-2} - \frac{1}{p+2} + \frac{3}{p-2} + \frac{p+3}{(p^2-4)}\ . \tag{24}$$

Next using the inverse Laplace transform of L2, L9, L10 and simplifying we get

$$\boxed{y(t) = -e^{-t} + 2e^{-2t}} \; . \tag{25}$$

6. MB 443.25 By using Laplace transforms, solve the following differential equation with given initial conditions:

$$y'' + 4y' + 5y = 2e^{-2t}\cos(t) \; , \quad y_0 = 0 \; , \quad y_0' = 3 \tag{26}$$

§**Solution**

We take the Laplace transformation of each term in Eq. 26

$$p^2Y - py_0 - y_0' - 4pY - 4y_0 + 4y_0 + 5Y = 2\frac{p+2}{(p+2)^2+1} \tag{27}$$

Now we plug in the values for the initial conditions and group terms:

$$(p^2 - 4p + 5)Y - 3 = 2\frac{p+2}{(p+2)^2+1} \; . \tag{28}$$

Next we solve for Y noting that $(p^2 - 4p + 5) = (p+2)^2 + 1$:

$$Y = 2\frac{p+2}{(p+2)^2+1} + \frac{3}{(p+2)^2-1} \; . \tag{29}$$

Next using the inverse Lapace transform of derived in MB 439.21 (Eq. 11) and L13 and simplifying we get

$$\boxed{y(t) = (3+t)e^{-2t}\sin(t)} \; . \tag{30}$$

7. MB 667.11 Find the real and imaginary parts $u(x, y)$ and $v(x, y)$ of the following function:

$$\frac{2z - i}{iz + 2} \; . \tag{31}$$

§**Solution**

First we expand Eq. 31 into rectangular coordinates:

$$\frac{2(x+iy) - i}{i(x+iy) + 2} \; . \tag{32}$$

Next we multiply the numerator and denomator by the complex conjugate of the denomator:

$$\frac{(2(x+iy) - i)(-ix - y + 2)}{(i(x+iy) + 2)(-ix - y + 2)} \; . \tag{33}$$

Now we expand the function into the real and imagainary parts:

$$\boxed{u(x,y) + iv(x,y) = \frac{3x}{x^2 + (y-2)^2} + i\frac{-2 - 2x^2 + 5y - 2y^2}{x^2 + (x-y)^2}} \; . \tag{34}$$

8. MB 672.24 Determine if $(y - ix)/(x^2 + y^2)$ is analytic.

§**Solution**

Since it is a function of the complex conjugate

$$C = \frac{y - ix}{x^2 + y^2} = \frac{-i(x + iy)}{zz^*} = \frac{-iz}{zz^*} = -\frac{i}{z^*} \ , \tag{35}$$

it is not analytic.

9. MB 673.37 Write the power series (about the origin) for the function $\tanh(z)$ and find the disk of convergence.

§**Solution**

The series for $\tanh(z)$ is

$$\begin{aligned} \tanh(z) = \frac{\sinh(z)}{\cosh(z)} &= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n}{dz^n} \tanh(z) \mid_{z=0} z^n \\ &= 0 + \frac{1}{1!} \operatorname{sech}^2(z)|_{z=0} z + \frac{1}{2!}(-2 \operatorname{sech}^2(z) \tanh^2(z)) z^2 + \cdots \\ &= z - \frac{z^3}{3} + \frac{2z^5}{15} - \cdots \end{aligned} \tag{36}$$

Note that

$$-i \tanh(iy) = -i \frac{e^{iy} - e^{-iy}}{e^{iy} + e^{-iy}} = \frac{\sin(y)}{\cos(y)} \ .$$

Since $\cos(\pi/2) = 0$, our nearest pole to the origin is $z = \pm i\pi/2$. Thus the radius of convergence is $r = |z| \leq \pi/2$.

10. MB 674.58 Show that the following function is harmonic, i.e., that they satisfy Laplace's equation. Find the function f(z) for which the given function is the real part $u(x, y)$. Show that $v(x, y)$ also satifies Laplaces's equation.

$$u(x, y) = \cosh(y) \cos(x) \tag{37}$$

§**Solution**

First we show that $u(x, y)$ satifies the Laplace equation:

$$\nabla^2 u = \frac{d^2}{dx^2} u(x, y) + \frac{d^2}{dy^2} u(x, y) = -\cosh(y) \cos(x) + \cosh(y) \cos(x) = 0 \ \checkmark$$

Using Cauchy-Riemann equations, we find

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = -\cosh(y) \sin(x) \ .$$

Thus if we integrate with respect to y we have

$$v(x, y) = -\sinh(y) \sin(x) + g(x)$$

where $g(x)$ must be restricted to an arbituary linear function of x in order to satify the Laplace equation. Additionally we find the $g(x)$ must be further restricted to a constant, C, in order to satify the Cauchy-Riemann equations. Hence,

$$\boxed{f(z) = \cosh(y)\cos(x) - i\sinh(y)\sin(x) + C = \cos(z) + C} \,.$$

Similarly,

$$\nabla^2 u = \frac{d^2}{dx^2}v(x,y) + \frac{d^2}{dy^2}v(x,y) = \sinh(y)\sin(x) - \sinh(y)\sin(x) = 0 \,. \checkmark$$