Mathematical Methods of Physics 116B- Spring 2018

Physics 116B

Home Work # 7 Solutions
Posted on May 24, 2018
Due in Class May 31, 2018

§**Required Problems: Each problem has 10 points**

E.g. MB 19.16 means problem #16 on page 19 in the book by M. Boas, 3rd Edition.

1. MB 677.4 Evaluate the line integral $\int dz/(1-z^2)$ along the whole positive imaginary axis.

   §**Solution**

$$\int_{iy=0}^{i\infty} dz \frac{1}{1-z^2} \tag{1}$$

   We begin by reexpressing Eq. 1 in rectangular coordinates $z = x + iy$ and $dz = dx + idy$:

$$\int_C \frac{dx + idy}{1-(x+iy)^2} = \int_C \frac{dx}{1-(x+iy)^2} + i\int_C \frac{dy}{1-(x+iy)^2} \tag{2}$$

   where C is the path. Since the path doesn't change along the x-direction, the first integral on the RHS Eq. 3 is zero and for the second integral x is a constant:

$$i\int_0^{\infty} \frac{dy}{1+y^2} = i\arctan(y)\Big|_0^{\infty} = \boxed{i\pi/2} \tag{3}$$

2. MB 677.11 Evaluate $\oint(\bar{z}-3)dz$ where C is the indicated closed curve along the first quadrant part of the circle $|z| = 2$, and the indicated parts of the x and y in Fig. 1.

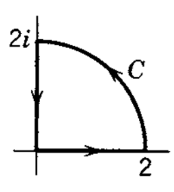

Figure 1: Contour

   §**Solution**

We split the contour into three parts $C = C_1 + C_2 + C_3$, where C_1 is the path along the x-axis, C_2 is the path along the arc, and C_3 is the path along the y-axis.

For the path along C_1 we transform to rectangular coordinates $z = x + iy$, $dz = dx + idy$ and set $y = 0$:

$$\int_{C_1} (\bar{z} - 3)dz = \int_{C_1} (x - iy)(dx + idy) = \Big(\frac{x^2}{2} - 3x\Big)\Big|_0^2 = -4 \; . \qquad (4)$$

Similarly, for the path along C_3 we transform to rectangular coordinates $z = x + iy$, $dz = dx + idy$ and set $x = 0$:

$$\int_{C_3} (\bar{z} - 3)dz = \int_2^0 \Big(-iy - 3\Big) = i\Big(\frac{-iy^2}{2} - 3y\Big)idy = -2 + 6i \; . \qquad (5)$$

For the path along C_2 we transform to polar coordinates $z = \rho e^{i\theta}$, $dz = d\rho e^{i\theta} + i\rho e^{i\theta}$ and set $\rho = 2$:

$$\begin{aligned} \int_{C_3} (\bar{z} - 3)dz &= 4i \int_0^{\pi/2} d\theta - 6i \int_0^{\pi/2} e^{i\theta} d\theta \\ &= 4i\theta\Big|_0^{\pi/2} - 6i\frac{e^{i\theta}}{i}\Big|_0^{\pi/2}(1+i) = 2\pi i - 6i + 6 \; . \end{aligned} \qquad (6)$$

Now we sum the results of the three integrals to get $\boxed{2\pi i}$.

$$(7)$$

3. MB 677.21 Differentiate Cauchy's formula to get

$$f'(z) = \frac{1}{2\pi i} \oint_C \frac{f(\omega)}{(\omega - z)^2} d\omega \; , \qquad (8)$$

and by differetiating n times, obtain

$$f'(z) = \frac{n!}{2\pi i} \oint_C \frac{f(\omega)}{(\omega - z)^{n+1}} d\omega \; . \qquad (9)$$

§**Solution**

Differenting Cauchy's formula gives

$$\begin{aligned} \frac{d}{dz} f(z) &= \frac{d}{dz}\frac{1}{2\pi i} \oint_C \frac{f(\omega)}{(\omega - z)} d\omega \\ &= -\frac{1}{2\pi i} \oint_C -(\omega - z)^{-2}(-1) f(\omega) d\omega \\ &= \frac{1}{2\pi i} \oint_C \frac{f(\omega)}{(\omega - z)^2} \end{aligned} \qquad (10)$$

where the additional (-1) comes from the chain rule. Similarly, differentiating Cauchy's formula n times gives

$$\frac{d^2}{dz^2}f(z) = \frac{1}{2\pi i}(-1)^2(-1)(-2)\oint_C (\omega - z)^2 f(\omega)d\omega$$
$$\frac{d^3}{dz^3}f(z) = \frac{1}{2\pi i}(-1)^3(-1)(-2)(-3)\oint_C (\omega - z)^3 f(\omega)d\omega$$
$$\vdots$$
$$\frac{d^n}{dz^n}f(z) = \frac{1}{2\pi i}(-1)^n(-1)(-2)(-3)\cdots(-n)\oint_C (\omega - z)^{n+1} f(\omega)d\omega \; .$$

Hence,

$$\frac{d^n}{dz^n}f(z) = \frac{n!}{2\pi i}\oint_C \frac{f(\omega)}{(\omega - z)^{n+1}}d\omega \; . \tag{11}$$

4. MB 686.14 Find the residue of the following function at the indicated points:

$$f(z) = \frac{1}{(3z+2)(2-z)} \quad \text{at } z = -2/3; \text{ and at } z = 2; \tag{12}$$

§**Solution**

The function $f(z)$ has pole a simple pole at $z = -2/3$, so corresponding residue given by

$$R(-2/3) = \lim_{z\to -2/3}(z + 2/3)f(z) = \lim_{z\to -2/3}\frac{-1}{3(z-2)} = \boxed{1/8} \, , \tag{13}$$

and there is a simple pole at $z = 2$ with corresponding residue of

$$R(2) = \lim_{z\to 2}(z-2)f(z) = \lim_{z\to 2}\frac{-1}{3(z+2/3)} = \boxed{-1/8} \, . \tag{14}$$

5. MB 687.20 Evaluate the residue of the following function at the indicated point:

$$f(z) = \frac{z}{1-z^4} \text{ at } z = i \; . \tag{15}$$

§**Solution**

First we factor the denominator of Eq. 15:

$$f(z) = \frac{-z}{(z-i)(z+i)(z-1)(z+1)} \tag{16}$$

We see that the function has a simple pole at $z = i$ and the residue is

$$R(i) = \lim_{z\to i}(z-i)f(z) = \lim_{z\to i}\frac{-z}{(z+i)(z-1)(z+1))} = \boxed{1/4} \tag{17}$$

6. MB 687.27 Evaluate the residue of the following function at the indicated point:

$$f(z) = \frac{\cos(z)}{1 - 2\sin(z)} \text{ at } z = \pi/6 \ . \tag{18}$$

§**Solution**

To find the residue of Eq. 20 we use L'Hospitals rule

$$R(\pi/6) = \lim_{z \to \pi/6} \frac{\cos(z)}{-2\cos(z)} = \boxed{-1/2} \ . \tag{19}$$

7. MB 687.35 Evaluate the residue of the following function at the indicated point:

$$f(z) = \frac{z}{(z^2+1)^2} \text{ at } z = i \ . \tag{20}$$

§**Solution**

We factor the denominator to find the poles:

$$f(z) = \frac{z}{(z+1)(z-1)} \tag{21}$$

We see that the function has pole of order to at $z = i$, so the residue is given by

$$R(i) = \lim_{z \to i} \frac{d}{dz}(z-i)f(z) = \lim_{z \to i} \left(\frac{1}{(z+i)^2} + \frac{-2z}{(z+i)^3} \right) = \boxed{0} \ . \tag{22}$$

8. MB 687.23 Evaluate the residue of the following function at the indicated point:

$$f(z) = \frac{e^{iz}}{9z^2+4} \text{ at } z = 2i/3 \ . \tag{23}$$

§**Solution**

We factor the denominator to find the poles:

$$f(z) = \frac{z}{9(z-2i/3)(z+2i/3)} \tag{24}$$

The pole at $z = 2i/3$ is a simple pole, hence the residue is

$$R(2i/3) = \lim_{z \to 2i/3} (z - 2i/3)f(z) = \lim_{z \to 2i/3} \frac{e^{iz}}{9(z+2i/3)} = \boxed{\frac{e^{-2/3}}{12i}} \tag{25}$$

9. MB 699.3 Evaluate the contour integral:

$$\int_0^{2\pi} \frac{d\theta}{5 - 4\sin(\theta)} \tag{26}$$

§**Solution**

This integral is equivalent to a contour integral over the unit circle. Note that $\sin\theta = (e^{i\theta} - e^{-i\theta})/2i$. Now if we set $z = e^{i\theta}$ such that $d\theta = dz/iz$ and $\sin\theta = (z - 1/z)/2i$, we obtain

$$\begin{aligned} \oint_C \frac{1}{5 - 4(z - 1/z)/2i} \frac{dz}{z} &= \int_C \frac{dz}{5z + 2i(z^2 - 1)} \\ &= -\frac{1}{2} \int_C \frac{dz}{(z - 2i)(z - i/2)} \ . \end{aligned} \tag{27}$$

To evaluate the integral we sum the residues contain in the unit circle

$$-\frac{1}{2} 2\pi i \sum_{z_p} R(z_p) \tag{28}$$

where z_p are the poles. Note that the residue at $z = 2i$ lies outside the unit circle. The residue at $z = i/2$ is

$$R(i/2) = \lim_{z \to i/2} (z - i/2) f(z) = -2/3i$$

where $f(z) = 1/(z - 2i)(z - i/2)$. Hence,

$$\int_0^{2\pi} \frac{d\theta}{13 + 5\sin(\theta)} = -\frac{1}{2} 2\pi i \sum_{z_p} R(z_p) = \boxed{2\pi/3} \ . \tag{29}$$

10. MB 699.4 Evaluate the contour integral:

$$\int_0^{2\pi} \frac{\sin^2\theta d\theta}{5 + 3\cos(\theta)} \ . \tag{30}$$

§**Solution**

As in the previous problem we notice that the integral is equivalent to the contour integral over the unit circle

$$\oint_C -\frac{(z - 1/z)^2/4}{5 + 3(z + 1/z)/2} \frac{dz}{iz} \ . \tag{31}$$

Next we factor the denominator and expand the numerator

$$-\frac{1}{6i} \oint_C \frac{z^2 - 1/z^2 - 2}{(z + 3)(z - 1/3)} = -\frac{1}{6i} \oint_C \frac{z^2 - 2}{(z + 3)(z + 1/3)} - \frac{1}{6i} \oint_C \frac{1}{z^2(z + 3)(z + 1/3)} \tag{32}$$

The first contour integral on the RSH of Eq. 32 has poles at $z = -1/3, -3$, however $z = -3$ lies outside the unit circle, and evaluating the integral using the residue method we obtain

$$-\frac{1}{6i}\oint_C \frac{z^2-2}{(z+3)(z+1/3)} = -\frac{1}{6i}2\pi i \sum_{z_p} R(z_p) = 17\pi/72\ . \qquad (33)$$

The second contour integral on the RSH of Eq. 32 has two poles that lie inside the unit circle, where the pole at $z = -1/3$ is first order and the poles at $z = 0$ is second order. We evaluate the integral using the residue method to find

$$-\frac{1}{6i}\oint_C \frac{z^2-2}{(z+3)(z+1/3)} = -\frac{1}{6i}2\pi i \sum_{z_p} R(z_p) = -\pi/72\ . \qquad (34)$$

Hence,

$$\int_0^{2\pi} \frac{\sin^2\theta d\theta}{5+3\cos(\theta)} = \boxed{2\pi/9}\ . \qquad (35)$$