Mathematical Methods of Physics 116B- Spring 2018

Physics 116B

Home Work # 8 Solutions
Posted on May 31, 2018
Due in Class May 6, 2018

§**Required Problems: Each problem has 10 points**

E.g. MB 19.16 means problem #16 on page 19 in the book by M. Boas, 3rd Edition.

1. MB 699.11 Evaluate the following using contour methods:

$$I = \int_0^\infty \frac{dx}{(4x^2+1)^3} \; . \tag{1}$$

§**Solution**

We can show that the integral in Eq. 1 can be found by computing the contour integral around the upper half plane since line integal along the arc vanishes:

$$2I = \oint_C \frac{dz}{(4x^2+1)^3} = \int_{-\infty}^\infty \frac{dz}{(4z^2+1)^3} = 2\int_0^\infty \frac{dx}{(4x^2+1)^3} \tag{2}$$

The simplest method to compute the contour integral is to find the residues contain in the countour. The first step is to factor the denominator and identify the poles z_p:

$$\oint_C \frac{dx}{(z-i/2)^3(z+i/2)^3} = 2\pi i \sum_{z_p} f(z_p) \tag{3}$$

There are poles at $z_p = -i/2, i/2$ both of order 3, but only the pole at $z_p = i/2$ is contained inside the contour. The residue at $z_p = i/2$ is

$$R(i/2) = \lim_{z_p \to i/2} \frac{1}{2!}\frac{d^2}{dz^2}(z-i/2)f(z) = -\frac{12i}{2\cdot 4^3} \; , \tag{4}$$

where $f(z) = 1/(4(z-i/2)(z+i/2))^3$. Hence,

$$I = 2\pi i \frac{-12i}{2\cdot 4^3} = \boxed{3\pi/32} \tag{5}$$

2. MB 700.30(a) Evaluate the following integral using the contour method:

$$I = \int_0^\infty \frac{dx}{1+x^4} \tag{6}$$

§**Solution**

The integral in Eq. 6 can be found by computing the contour integral around the upper half plane since the line integral along the arc vanishes:

$$2I = \oint_C \frac{dz}{1+z^4} = \int_{-\infty}^{\infty} \frac{dx}{1+x^4} = 2\int_0^{\infty} \frac{dx}{1+x^4} \ . \tag{7}$$

We compute the contour integral using the residue method. We begin by factoring the denominator and finding the poles

$$\oint_C \frac{dz}{(z+ie^{i\pi/4})(z-ie^{i\pi/4})(z-e^{i\pi/4})(z+e^{i\pi/4})} = 2\pi i \sum_{z_p} f(z_p) \tag{8}$$

We see there are poles at $z_p = e^{i3\pi/4}, e^{i\pi/4}, -e^{i3\pi/4}, -e^{i\pi/4}$, yet only the pole at $z_p = e^{i3\pi/4}, e^{i\pi/4}$ are located inside the contour. The residues at $z_p = e^{i3\pi/4}$ and $z_p = e^{i\pi/4}$ are

$$\begin{aligned} R(e^{i3\pi/4}) &= \lim z \to e^{i3\pi/4}(z - e^{i3\pi/4}) f(z) = -1/4e^{i3\pi/8} \ , \\ R(e^{i\pi/4}) &= \lim z \to e^{i\pi/4}(z - e^{i3\pi/4}) f(z) = -1/4e^{i\pi/8} \ , \end{aligned} \tag{9}$$

respectively and $f(z) = 1/(1+z^4)$. Hence, the integral is

$$I = \frac{2\pi i}{2} \sum_{z_p} f(z_p) = \frac{\pi}{2\sqrt{2}} \tag{10}$$

3. MB 701.42 If $F(z) = f'(z)/f(z)$,

   a) show that the residue of $F(z)$ at an nth order zero of $f(z)$, is n.

   b) Also show that the reside of $F(z)$ at a pole of order $-p$ of $f(z)$, is -p.

§**Solution**

a) If $f(z)$ has a zero at $z = a$ of order n, then the function is characterized by the series

$$f(z) = a_n(z-a)^n + a_{n+1}(z-a)^{n+1} + \cdots \ . \tag{11}$$

Therefore

$$F(z) = \frac{a_n n(z-a)^{n-1} + a_{n+1}(n+1)(z-a)^n + \cdots}{a_n(z-a)^n + a_{n+1}(z-a)^{n+1} + \cdots} \ . \tag{12}$$

In the limit as we approach $z = a$, we obtain

$$F(z) = \frac{a_n n}{a_n(z-a)} = \frac{-p}{(z-a)} \ . \tag{13}$$

Hence, the residue of the function is $\boxed{n}$.

b) Similarly, If $f(z)$ has a pole at $z = a$ of order p, then the function is characterized by the series

$$f(z) = b_p(z-a)^{-p} + b_{p-1}(z-a)^{p-1} + \cdots + b_1(z-1)^{-1} + a_0 + a_1(z-a)\cdots \ . \tag{14}$$

Therefore

$$F(z) = \frac{b_p(-p)(z-a)^{-p-1} + b_{p-1}(-p+1)(z-a)^{-p+1} + \cdots + b_1(-1)(z-1)^{-2} + a_1\cdots}{b_p(z-a)^{-p} + b_{p-1}(z-a)^{-p+1} + \cdots + b_1(z-1)^{-1} + a_0 + a_1(z-a)\cdots} \ . \tag{15}$$

In the limit as z approaches a, we obtain

$$F(z) = \frac{b_p(-p)}{b_p(z-a)} = \frac{-p}{(z-a)} \ . \tag{16}$$

This is easy to see if we multiply the numerator and denominator by $(z-a)^p$. Hence, the residue of the function is $\boxed{-p}$.

4. MB 681.8 Find the Laurent series about the origin and find the residue at the origin for the following function:

$$f(z) = \frac{30}{(z+1)(z-2)(z+3)} \ . \tag{17}$$

§**Solution**

To find the Laurent series, it is best to split the function into partical fractions:

$$f(z) = \frac{2}{z-2} - \frac{5}{z+1} + \frac{3}{z+3} \ . \tag{18}$$

We see that $f(z)$ has poles at $z = -1, 2, -3$ therefore for a series about the origin there will be annular rings at a radius of $|z| = 1, 2, 3$ respectively. There for four regions of interest: $0 < |z| < 1$, $1 < |z| < 2$, $2 < |z| < 3$, and $|z| > 3$. Next we can convert the partial fractions into power series by writing them in the form a geometric series:

$$-\frac{5}{z+1} = -5\sum_{n=0}^{\infty}(-1)^n z^n \qquad \text{for} \quad |z| < 1 \tag{19}$$

$$-\frac{1}{z}\frac{5}{1+1/z} = -5\sum_{n=0}^{\infty}(-1)^n \left(\frac{1}{z}\right)^{n+1} \qquad \text{for} \quad |z| > 1 \tag{20}$$

$$-\frac{1}{1-z/2} = -\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n \qquad \text{for} \quad |z| < 2 \tag{21}$$

$$\frac{2}{z}\frac{1}{1-2/z} = \sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^{n+1} \qquad \text{for} \quad |z| > 2 \tag{22}$$

$$\frac{1}{1+z/3} = \sum(-1)^n(z/3)^n \qquad \text{for} \quad |z| < 3 \tag{23}$$

$$\frac{3}{z}\frac{1}{1+3/z} = \sum_{n=0}^{\infty}(-1)^n(3/z)^{n+1} \qquad \text{for} \quad |z| > 3 \tag{24}$$

We can see that the series in the region for $0 < |z| < 1$ has no term $1/z$, therefore b_1, the coefficient corresponding to the residue must be zero, i.e. $\boxed{R(0) = 0}$.

5. MB 686.5 Find the Laurent series and the residue at the indicated point.

$$f(z) = \frac{e^z}{z^2+1} \text{ at } z = 1\ . \tag{25}$$

§**Solution** If we break this function up into partial fractions, we obtain

$$f(z) = \frac{e^z}{2(z-1)} - \frac{e^z}{2(z+1)}\ . \tag{26}$$

There are poles at $z_p = -1,\ 1$ and ∞, such that there are three annular rings separating the regions of convergence about the point $z_0 = 1$: $0 < |z-1| < 2$, $2 < |z-1| < \infty$ and $|z| > \infty$. In order to find the residue at $z = 1$ we examine the Laurent series in powers of $(z-1)$ in the region $0 < |z-1| < 2$. Now we want to find the power series expansion of the functions in Eq. 26 that are convergent in this region:

$$e^z = e^{z-1}e = e\sum_{n=0}^{\infty}\frac{(z-1)^n}{n!} \qquad \forall \quad |z| < \infty \tag{27}$$

$$\frac{1}{z+1} = \frac{1}{2(1+(z-1)/2)} = \sum_{n=0}^{\infty}(-1)^n\frac{(z-1)^n}{2^n} \qquad \forall \quad |z| < |2|\ . \tag{28}$$

The Laurent series is

$$\begin{aligned} f(z) = {} & \frac{e}{2}\frac{1}{z-1}\sum_{n=0}^{\infty}\frac{(z-1)^n}{n!} \\ & + \frac{e}{4}\left(\sum_{n=0}^{\infty}\frac{(z-1)^n}{n!}\right)\left(\sum_{n=0}^{\infty}(-1)^n\frac{(z-1)^n}{2^n}\right) \end{aligned} \tag{29}$$

and the residue is the coefficient, b_1, attached to the $1/(z-1)$ power and hence the residue is $R(1) = \boxed{e/2}$.

6. MB 687.27 Find the residue of the following function at the indicated point:

$$f(z) = \frac{\cosh(z) - 1}{z} \text{ at } z = 0\ . \tag{30}$$

§**Solution** We see that there is a pole at $z = 0$, but the question is what is the order of the pole? We could guess and check the order using the L'Hospital method, or we may expand $\cosh z$ as a power series about $z_0 = 0$:

$$f(z) = \frac{\sum_{n=0}^{\infty}\frac{z^{2n}}{2n!} - 1}{z^7} = \sum_{n=1}^{\infty}\frac{z^{2n-7}}{2n!} \tag{31}$$

By examining the principle part of the series we see there is pole at $z = 0$ of order 7 and the residue is $b_1 = \boxed{1/6!}$ which is just the coefficent of the $1/z$ term.

7. MB 564.5 Find the two solutions to the following the differential equation, $L[y]$, using the power series method and verify:

$$L[y] = y'' - y = 0 \tag{32}$$

§**Solution** We start with the ansatz that

$$\begin{aligned} y &= \sum_{n=0}^{\infty} c_n x^n \\ y' &= \sum_{n=0}^{\infty} c_n n x^{n-1} \\ y'' &= \sum_{n=0}^{\infty} c_n n(n-1)x^{n-2} \end{aligned} \tag{33}$$

where c_n are real or complex coefficients. Next we insert Eq. 33 into Eq. 32:

$$L[y] = \sum_{n=0}^{\infty} c_n n(n-1)x^{n-2} - \sum_{n=0}^{\infty} c_n x^n \tag{34}$$

Our next step is to rewrite the first summation so that the exponent is n instead of $n-2$. This is accomplished by the subsitution $n \to n+2$:

$$L[y] = \sum_{n=0}^{\infty} c_{n+2}(n+1)(n+2)x^n - \sum_{n=0}^{\infty} c_n x^n \tag{35}$$

Now we set the sum of the coeffiencts of like powers of x equal to zero:

$$c_{n+2}(n+1)(n+2) - c_n = 0; . \tag{36}$$

Equation (36) is a recurrence formula that produces all the coefficients. To find the two solutions, we provide two choices for c_0 and c_1. The simplest possible choices are (i) $c_0 = 1$, $c_1 = 0$; (ii) $c_0 = 0$ and $c_1 = 1$. We find for (i) and (ii): $\boxed{y = \sum_{n=0}^{\infty} x^{2n}/(2n!) = \cos(x)}$ and $\boxed{y = \sum_{n=0}^{\infty} x^{2n-1}/(2n-1)!) = \sin(x)}$ respectively.

8. MB 564.7 Find the two solutions to the following the differential equation, $L[y]$, using the power series method and verify:

$$L[y] = x^2 y'' - 3xy + 3y = 0 \tag{37}$$

§**Solution**

We start with the ansatz written out in Eq. (33) and we insert that into Eq. 37:

$$\begin{aligned} L[y] &= x^2 \sum_{n=0}^{\infty} c_n n(n-1)x^{n-2} - 3x \sum_{n=0}^{\infty} c_n n x^{n-1} + 3 \sum_{n=0}^{\infty} c_n x^n \\ &= \sum_{n=0}^{\infty} c_n n(n-1)x^n - 3 \sum_{n=0}^{\infty} c_n n x^n + 3 \sum_{n=0}^{\infty} c_n x^n \end{aligned} \tag{38}$$

where we pulled the x terms into the summations. Our next step is to write the sum coefficients of like powers of x and set them equal to zero:

$$c_n(n)(n-1) - 3(c_n - n) = 0; . \tag{39}$$

Upon examination we see that $n = 3$ is a solutions of Eq. 39 where all other coefficients are zero and there is a second solution for $n = 1$ where all other coefficients are zero and hence, $\boxed{y = c_1 t + c_3 t^3\ .}$

9. MB 578.4 Show that $\int_{-1}^{1} dx P_\ell(x) P'_{\ell-1}(x) = 0$ using the relation

$$\int_{-1}^{1} dx P_\ell(x)(\text{any polynomial of degree} < \ell) = 0\ . \tag{40}$$

And also show that $\int_{-1}^{1} dx P'_\ell(x) P_{\ell+1}(x) = 0$.

§**Solution** The solution to this is actually very simple. We note that $P_\ell(x)$ is a polynomial of degree ℓ, such that $P'_\ell(x)$ is a polynomial of degree $\ell - 1$. Hence, using Eq. 40 we find $\int_{-1}^{1} dx P_\ell(x) P'_{\ell-1}(x) = 0$ because $\ell - 2 < \ell$. Similarly, $\int_{-1}^{1} dx P'_\ell(x) P_{\ell+1}(x) = 0$ because $\ell - 1 < \ell + 1$.

10. MB 582.10 Expand the polynomial, $f(x) = 3x^2 + x - 1$, in a Legendre series.

§**Solution** To expand a function as a Legendre series we must determine the coefficients of this series

$$\sum_{\ell=0}^{\infty} c_\ell P_\ell(x) \; . \tag{41}$$

This is accomplished using the orthogonality relation

$$\int_{-1}^{1} dx f(x) P_m(x) = \sum_{\ell=0}^{\infty} c_\ell \int_{-1}^{1} dx P_\ell(x) P_m(x) = c_m \frac{2}{2m+1} \tag{42}$$

Essentially, we evaluate the integral on the LHS of Eq. 42 for $P_0(x)$, $P_1(x), \ldots$ in order to determinte the coefficents:

$$c_m = \frac{2m+1}{2} \int_{-1}^{1} dx f(x) P_m(x) \tag{43}$$

where $c_0 = 0$, $c_1 = 1$, $c_2 = 2$ and $c_m = 0$ for all $m > 3$ according to Eq. 40. Hence, the Legendre series is

$$\sum_{\ell=0}^{\infty} c_\ell P_\ell(x) = P_1(x) + 2P_2(x) \; . \tag{44}$$