PHYS 116C
Homework One

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Einstein Summation Notation

I get very bored writing summation symbol over and over. Therefore, I will be using Einstein Summation Notation throughout this document. Einstein summation notation is very simple to use. Every time you see repeated indices in a given term, a summation symbol over that index is implied. That is

$$a_{ij}v_j \implies \sum_j a_{ij}v_j \tag{0.1}$$

or

$$c_{il} = b_{ijkl}v_{jk} \implies c_{il} = \sum_{j,k} a_{ijkl}v_{jk} \tag{0.2}$$

or

$$a = b_{ii} \implies a = \sum_i b_{ii} \tag{0.3}$$

We will also be ignoring the difference between upper and lower indices. We will write all indices as lower indices.

1 Problem One

[**Boas, Ch.10, Sec.2, Problem 2**] Show that the sum of the squares of the direction cosines of a line through the origin is equal to 1 *Hint*: Let (a, b, c) be a point on the line at distance 1 from the origin. Write the direction cosines in terms of (a, b, c).

1.1 Solution One

Let the line be defined by

$$\boldsymbol{\ell}(t) = t(a\hat{i} + b\hat{j} + c\hat{k}) \tag{1.1}$$

where $t \in \mathbb{R}$. Let the direction cosines be c_1, c_2 and c_3, given by

$$\boldsymbol{\ell}(t) \cdot \hat{i} = |\boldsymbol{\ell}(t)|c_1, \qquad \boldsymbol{\ell}(t) \cdot \hat{j} = |\boldsymbol{\ell}(t)|c_2, \qquad \boldsymbol{\ell}(t) \cdot \hat{k} = |\boldsymbol{\ell}(t)|c_3. \tag{1.2}$$

where $|\boldsymbol{\ell}(t)| = t\sqrt{a^2 + b^2 + c^2}$. Then the sum of the squares of the directional cosines is:

$$c_1^2 + c_2^2 + c_3^2 = \frac{(\boldsymbol{\ell}(t) \cdot \hat{i})^2 + (\boldsymbol{\ell}(t) \cdot \hat{j})^2 + (\boldsymbol{\ell}(t) \cdot \hat{j})^2}{|\boldsymbol{\ell}(t)|^2} = \frac{a^2t^2 + b^2t^2 + c^2t^2}{t^2(a^2 + b^2 + c^2)} = 1 \tag{1.3}$$

2 Problem Two

[Boas, Ch.10, Sec.2, Problem 3] Consider the matrix A in (2.7) or (2.10). Think of the elements in each row (or column) as the components of a vector. Show that the row vectors form an orthonormal triad (that is each is of unit length and they are all mutually orthogonal), and the column vectors form an orthonormal triad.

2.1 Solution Two

Consider two coordinate systems: one with unit orthonormal unit vectors $\hat{\imath}, \hat{\jmath}$ and $\hat{k}$ and the second with orthonormal unit vectors $\hat{\imath}', \hat{\jmath}'$ and $\hat{k}'$. If we have a vector $\boldsymbol{V}$ written in terms of the unprimed coordinates

$$\boldsymbol{V} = V_1\hat{\imath} + V_2\hat{\jmath} + V_3\hat{k} \tag{2.1}$$

Then the same vector can be written in the new coordinates:

$$\boldsymbol{V} = V_1'\hat{\imath}' + V_2'\hat{\jmath}' + V_3'\hat{k}' \tag{2.2}$$

with

$$V_i = A_{ij}V_j \tag{2.3}$$

$$\boldsymbol{A} = \begin{pmatrix} \hat{\imath}\cdot\hat{\imath}' & \hat{\jmath}\cdot\hat{\imath}' & \hat{k}\cdot\hat{\imath}' \\ \hat{\imath}\cdot\hat{\jmath}' & \hat{\jmath}\cdot\hat{\jmath}' & \hat{k}\cdot\hat{\jmath}' \\ \hat{\imath}\cdot\hat{k}' & \hat{\jmath}\cdot\hat{k}' & \hat{k}\cdot\hat{k}' \end{pmatrix} \tag{2.4}$$

Let's form vectors out of the rows of $\boldsymbol{A}$ and consider the vector to be in the unprimed coordinate system. Call these vectors $\boldsymbol{r}_i$. For example:

$$\boldsymbol{r}_1 = (\hat{\imath}\cdot\hat{\imath}')\hat{\imath} + (\hat{\jmath}\cdot\hat{\imath}')\hat{\jmath} + (\hat{k}\cdot\hat{\imath}')\hat{k} \tag{2.5}$$

$$\boldsymbol{r}_2 = (\hat{\imath}\cdot\hat{\jmath}')\hat{\imath} + (\hat{\jmath}\cdot\hat{\jmath}')\hat{\jmath} + (\hat{k}\cdot\hat{\jmath}')\hat{k} \tag{2.6}$$

$$\boldsymbol{r}_3 = (\hat{\imath}\cdot\hat{k}')\hat{\imath} + (\hat{\jmath}\cdot\hat{k}')\hat{\jmath} + (\hat{k}\cdot\hat{k}')\hat{k} \tag{2.7}$$

One immediately recognizes that $\boldsymbol{r}_1 = \hat{\imath}'$, $\boldsymbol{r}_2 = \hat{\jmath}'$ and $\boldsymbol{r}_3 = \hat{k}'$. Be definition, $\hat{\imath}', \hat{\jmath}$ and $\hat{k}'$ are of unit length and orthogonal and hence, form an orthonormal triad. To prove the columns form an orthonormal triad, we consider the column vectors as vectors in the primed coordinate system, i.e. :

$$\boldsymbol{c}_1 = (\hat{\imath}\cdot\hat{\imath}')\hat{\imath}' + (\hat{\imath}\cdot\hat{\jmath}')\hat{\jmath}' + (\hat{\imath}\cdot\hat{k}')\hat{k}' \tag{2.8}$$

and so forth. Then, one can see that $\boldsymbol{c}_1 = \hat{\imath}$, $\boldsymbol{c}_2 = \hat{\jmath}$ and $\boldsymbol{c}_3 = \hat{k}$, which form an orthonormal triad.

3 Problem Three

[Boas, Ch.10, Sec.2, Problem 7] Following what we did in equations (2.14) to (2.17), show that the direct product of a vector and a 3rd-rank tensor is a 4th-rank tensor. Also show that the direct product of two 2nd-rank tensors is a 4th-rank tensor. Generalize this to show that the direct product of two tensors of ranks m and n is a tensor of rank $m+n$.

3.1 Solution Three

Let $\boldsymbol{V}$ be a vector (rank-1 tensor) and $\boldsymbol{T}$ be a rank-3 tensor. Consider the product of the two. Let the product be $\boldsymbol{S}$. Then

$$S_{ijkl} = V_i T_{jkl} \tag{3.1}$$

To show that $\boldsymbol{S}$ is a rank-4 tensor, we need to show that $\boldsymbol{S}$ transforms as a rank-4 tensor. Since $\boldsymbol{V}$ and $\boldsymbol{T}$ are tensors, they transform as

$$V'_i = a_{ij} V_j \tag{3.2}$$

$$T'_{ijk} = a_{il} a_{jm} a_{kn} T_{lmn} \tag{3.3}$$

Therefore, we can see that $\boldsymbol{S}$ transforms as

$$S'_{ijkl} = V'_i T'_{jkl} \tag{3.4}$$

$$= (a_{im} V_m)(a_{jn} a_{kp} a_{lq} T_{npq}) \tag{3.5}$$

$$= a_{im} a_{jn} a_{kp} a_{lq} S_{mnpq} \tag{3.6}$$

Hence, we can see that $\boldsymbol{S}$ transforms as a rank-4 tensor. Next, let $\boldsymbol{A}$ and $\boldsymbol{B}$ be rank-2 tensors. Let $\boldsymbol{C} = \boldsymbol{AB}$ (or $C_{ijkl} = A_{ij} B_{kl}$). Let's show that $\boldsymbol{C}$ transforms as a rank-4 tensor:

$$C'_{ijkl} = A'_{ij} B'_{kl} \tag{3.7}$$

$$= a_{in} a_{jm} A_{nm} a_{kp} a_{lq} B_{pq} \tag{3.8}$$

$$= a_{in} a_{jm} a_{kp} a_{lq} C_{nmpq} \tag{3.9}$$

Next, let $\boldsymbol{A}$ be a rank-n tensor and $\boldsymbol{B}$ be a rank-m tensor. We would like to show that the product $\boldsymbol{C} = \boldsymbol{AB}$ is a rank-$(n+m)$ tensor. Then, $\boldsymbol{C}$ transforms as

$$C'_{i_1 \dots i_{n+m}} = A'_{i_1 \dots i_n} B'_{i_{n+1} \dots i_{n+m}} \tag{3.10}$$

$$= \left(a_{i_1 j_1} \cdots a_{i_n j_n} A_{i_1 \dots i_n}\right) \left(a_{i_{n+1} j_{n+1}} \cdots a_{i_{n+m} j_{n+m}} B_{i_{n+1} \dots i_{n+m}}\right) \tag{3.11}$$

$$= a_{i_1 j_1} \cdots a_{i_{n+m} j_{n+m}} A_{i_1 \dots i_n} B_{i_{n+1} \dots i_{n+m}} \tag{3.12}$$

which is how a rank-$(n+m)$ tensor transforms.

4 Problem Four

[Boas, Ch.10, Sec.3, Problem 2] Show that the fourth expression in (3.1) is equal to $\partial u/\partial x'_i$. By equations (2.6) and (2.10), show that $\partial x_j/\partial x'_i = a_{ij}$, so

$$\frac{\partial u}{\partial x'_i} = \frac{\partial u}{\partial x_j}\frac{\partial x_j}{\partial x'_i} = a_{ij}\frac{\partial u}{\partial x_j} \tag{4.1}$$

Compare this with equation (2.12) to show that ∇u is a Cartesian vector. Hint: Watch the summation indices carefully and if it helps, put back the summation signs or write sums out in detail as in (3.1) until you get used to summation convention.

4.1 Solution Four

Let $u = u(x, y, z)$ where $x = x(x', y', z'), y = y(x', y', z')$ and $z = z(x', y', z')$. Then, by the chain rule,

$$\frac{\partial u}{\partial x'} = \frac{\partial x}{\partial x'}\frac{\partial u}{\partial x} + \frac{\partial y}{\partial x'}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial x'}\frac{\partial u}{\partial z} = \frac{\partial x_j}{\partial x'_1}\frac{\partial u}{\partial x_j} \tag{4.2}$$

where we've defined $x, y, z = x_1, x_2, x_3$ and $x', y', z' = x'_1, x'_2, x'_3$. Repeating this for x'_2 and x'_3, we find that

$$\frac{\partial u}{\partial x'_i} = \frac{\partial x_1}{\partial x'_i}\frac{\partial u}{\partial x_1} + \frac{\partial x_2}{\partial x'_i}\frac{\partial u}{\partial x_2} + \frac{\partial x_3}{\partial x'_i}\frac{\partial u}{\partial x_2} = \frac{\partial x_j}{\partial x'_i}\frac{\partial u}{\partial x_j} \tag{4.3}$$

Given that

$$x'_i = a_{ij}x_j \qquad \Longrightarrow \qquad x_i = x'_j a_{ji} \tag{4.4}$$

we can see that

$$\frac{\partial x_i}{\partial x'_j} = a_{ji} \qquad \text{or} \qquad \frac{\partial x_j}{\partial x'_i} = a_{ij} \tag{4.5}$$

Therefore, we have that

$$\frac{\partial u}{\partial x'_i} = \frac{\partial x_j}{\partial x'_i}\frac{\partial u}{\partial x_j} = a_{ij}\frac{\partial u}{\partial x_j} \tag{4.6}$$

Therefore, we can see that $\partial u/\partial x_i$ is a rank-1 tensor or a vector. Consider ∇u. This is defined as

$$\nabla u = \frac{\partial u}{\partial x}\hat{i} + \frac{\partial u}{\partial y}\hat{j} + \frac{\partial u}{\partial z}\hat{k} \tag{4.7}$$

Hence, the i^{th} component of ∇u is

$$(\nabla u)_i = \frac{\partial u}{\partial x_i} \tag{4.8}$$

Since we've shown that $\partial u/\partial x_i$ is a vector, we can see that ∇u is a vector.

5 Problem Five

[Boas, Ch.10, Sec.3, Problem 5] Show that $T_{ijklm}S_{lm}$ is a tensor and find its rank (assuming that $\boldsymbol{T}$ and $\boldsymbol{S}$ are tensors of the rank indicated by the indices).

5.1 Solution Five

Since $\boldsymbol{T}$ and $\boldsymbol{S}$ are tensors or rank 5 and 2, we know that

$$T'_{i_1i_2i_3i_4i_5} = a_{i_1j_1}a_{i_2j_2}a_{i_3j_3}a_{i_4j_4}a_{i_5j_5}T_{j_1j_2j_3j_4j_5} \tag{5.1}$$

$$S'_{i_1i_2} = a_{i_1j_1}a_{i_2j_2}S_{j_1j_2} \tag{5.2}$$

Define $\boldsymbol{X} = \boldsymbol{TS}$. Then, we can see that

$$X'_{i_1i_2i_3} = T'_{i_1i_2i_3i_4i_5}S'_{i_4i_5} = a_{i_1j_1}a_{i_2j_2}a_{i_3j_3}a_{i_4j_4}a_{i_5j_5}a_{i_4k_4}a_{i_5k_5}T_{j_1j_2j_3j_4j_5}S_{k_4k_5} \tag{5.3}$$

Note that

$$a_{i_4j_4}a_{i_4k_4} = \left(\boldsymbol{A}^T\right)_{j_4i_4}(\boldsymbol{A})_{i_4k_4} = \left(\boldsymbol{A}^T\boldsymbol{A}\right)_{j_4k_4} = \delta_{j_4k_4} \tag{5.4}$$

where we used the fact that $\boldsymbol{A}$ is orthogonal ($\boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{I}$). Using this and $a_{i_5j_5}a_{i_5k_5} = \delta_{j_5k_5}$, we can see that

$$X'_{i_1i_2i_3} = T'_{i_1i_2i_3i_4i_5}S'_{i_4i_5} \tag{5.5}$$

$$= a_{i_1j_1}a_{i_2j_2}a_{i_3j_3}\delta_{j_4k_4}\delta_{j_5k_5}T_{j_1j_2j_3j_4j_5}S_{k_4k_5} \tag{5.6}$$

$$= a_{i_1j_1}a_{i_2j_2}a_{i_3j_3}T_{j_1j_2j_3j_4j_5}S_{j_4j_5} \tag{5.7}$$

$$= a_{i_1j_1}a_{i_2j_2}a_{i_3j_3}X_{j_1j_2j_3} \tag{5.8}$$

Hence, $\boldsymbol{X}_{ijk} = T_{ijklm}S_{lm}$ is a rank-3 tensor.

6 Problem Six

[**Boas, Ch.10, Sec.3, Problem 9**] Prove the quotient rule for

$$X_i A_{ij} = B_j, \tag{6.1}$$

that is, given $\boldsymbol{X}\boldsymbol{A} = \boldsymbol{B}$ where $\boldsymbol{A}$ is any arbitrary tensor and $\boldsymbol{B}$ is a non-zero tensor, show that $\boldsymbol{X}$ is a tensor. Hints: Follow the general method in (3.6) to (3.9). See the last sentence of the section.

6.1 Solution Six

We know that $\boldsymbol{X}$ and $\boldsymbol{B}$ are tensors. Therefore,

$$B'_i = a_{ij} B_j \tag{6.2}$$
$$A'_{ij} = a_{im} a_{jn} A_{mn} \tag{6.3}$$

Using these, we find that

$$B'_j = a_{jm} B_m = X'_i A'_{ij} = X'_i a_{im} a_{jn} A_{mn} \tag{6.4}$$

Thus,

$$a_{jm} B_m - X'_i a_{im} a_{jn} A_{mn} = 0 \tag{6.5}$$

Using $X_i A_{ij} = B_j$ we have

$$a_{jm} X_n A_{nm} - X'_i a_{im} a_{jn} A_{mn} = 0 \tag{6.6}$$

Relabeling indices and factoring out the tensor $\boldsymbol{A}$, we find

$$\left(X_m - X'_i a_{im}\right) a_{jn} A_{mn} = 0 \tag{6.7}$$

Since $\boldsymbol{A}$ is an arbitrary tensor and $\boldsymbol{B}$ is non-zero (hence $\boldsymbol{X}$ is non-zero), we require that

$$X_m = X'_i a_{im} \qquad \Longrightarrow \qquad \boldsymbol{X} = \boldsymbol{a}^T \boldsymbol{X}' \tag{6.8}$$

Given that $\boldsymbol{a}\boldsymbol{a}^T = \boldsymbol{I}$, we can see that $\boldsymbol{X}' = \boldsymbol{a}\boldsymbol{X}$ or $X'_i = a_{ij} X_j$. Therefore, $\boldsymbol{X}$ transforms like a tensor.

7 Problem Seven

[Boas, Ch.10, Sec.4, Problem 4] Find the inertia tensor about the origin for a mass of uniform density = 1, inside the part of the unit sphere where $x > 0$, $y > 0$, and find the principal moments of inertia and the principal axes. Note that this is similar to Example 5 but the mass is both above and below the (x, y) plane. Warning hint: This time don't make the assumptions about symmetry that we did in Example 5.

7.1 Solution Seven

We know that the angular momentum of an infinitesimal mass of the object is given by

$$d\boldsymbol{L} = dm\ \boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{r}) \tag{7.1}$$

where $dm = dV = r^2 \sin\theta dr d\theta d\phi$ and

$$\boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{r})_x = -xy\omega_y + z(\omega_x z - x\omega_z) + y^2\omega_x \tag{7.2}$$
$$\boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{r})_y = x^2\omega_y - xy\omega_x + z(\omega_y z - y\omega_z) \tag{7.3}$$
$$\boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{r})_z = \omega_z \left(x^2 + y^2\right) - z(x\omega_x + y\omega_y) \tag{7.4}$$

Using

$$x = r\sin\theta\cos\phi \tag{7.5}$$
$$y = r\sin\theta\sin\phi \tag{7.6}$$
$$z = r\cos\theta \tag{7.7}$$

and integrating over $\pi \leq \theta \leq 0$, $\pi/2 \leq \phi \leq 0$ and $1 \leq r \leq 0$, we find

$$\boldsymbol{L}_x = \frac{2}{15}(\pi\omega_x - \omega_y) \tag{7.8}$$
$$\boldsymbol{L}_y = \frac{2}{15}(\pi\omega_y - \omega_x) \tag{7.9}$$
$$\boldsymbol{L}_z = \frac{2\pi\omega_z}{15} \tag{7.10}$$

Using the definition $\boldsymbol{L} = \boldsymbol{I}\boldsymbol{\omega}$, we determine that

$$\boldsymbol{I} = \frac{2}{15}\begin{pmatrix} \pi & -1 & 0 \\ -1 & \pi & 0 \\ 0 & 0 & \pi \end{pmatrix} \tag{7.11}$$

The eigenvectors and eigen values are

$$v_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \qquad \lambda_1 = \frac{2(1+\pi)}{15} \tag{7.12}$$

$$v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \lambda_2 = \frac{2\pi}{15} \tag{7.13}$$

$$v_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad \lambda_3 = \frac{2(\pi-1)}{15} \tag{7.14}$$

8 Problem Eight

[**Boas, Ch.10, Sec.4, Problem 6**] For the mass distribution consisting of a point masses 1 and 2 located at $(1, 1, -2)$ and $(1, 1, 1)$ respectively, find the inertia tensor about the origin, and find the principal moments of inertia and the principal axes.

8.1 Solution Eight

To determine $\boldsymbol{L}$ we need to evaluate Eq. (7.2) at $(1, 1, -2)$ and $(1, 1, 1)$. Summing the two results and multiplying the second by 2, we find

$$\boldsymbol{L}_x = 9\omega_x - 3\omega_y \tag{8.1}$$
$$\boldsymbol{L}_y = 9\omega_y - 3\omega_x \tag{8.2}$$
$$\boldsymbol{L}_z = 6\omega_z \tag{8.3}$$

The moment of inertia tensor is thus

$$\boldsymbol{I} = \begin{pmatrix} 9 & -3 & 0 \\ -3 & 9 & 0 \\ 0 & 0 & 6 \end{pmatrix} \tag{8.4}$$

The eigenvalues and eigenvectors are

$$v_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \qquad \lambda_1 = 12 \tag{8.5}$$
$$v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \lambda_2 = 12 \tag{8.6}$$
$$v_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad \lambda_3 = 6 \tag{8.7}$$

9 Problem Nine

[Boas, Ch.10, Sec.5, Problem 3] Show that $\delta_{ij}\epsilon_{klm}$ is an isotropic tensor of rank 5. Hint: Combine equations (5.4) and (5.7).

9.1 Solution Nine

There is not much work to be done here. From problem three, we know that $\delta_{ik}\epsilon_{klm}$ is a tensor of rank 5. Next, we show it is isotropic:

$$\delta'_{ij}\epsilon'_{klm} = a_{in}a_{jq}\delta_{nq}a_{kr}a_{ls}a_{mt}\epsilon_{rst} = a_{in}a_{jn}\epsilon_{klm} = \delta_{ij}\epsilon_{klm} \tag{9.1}$$

where we used $a_{in}a_{jn} = \left(\boldsymbol{a}\boldsymbol{a}^T\right)_{ij} = \delta_{ij}$ and $\epsilon_{\alpha\beta\gamma}\det(\boldsymbol{a}) = a_{\alpha i}a_{\beta j}a_{\gamma k}\epsilon_{ijk}$ (as well as $\det(\boldsymbol{a}) = 1$).

10 Problem Ten

[Boas, Ch.10, Sec.5, Problem 12] Write and prove in tensor notation:

(c) Lagrange's identity:

$$(\boldsymbol{A} \times \boldsymbol{B}) \cdot (\boldsymbol{C} \times \boldsymbol{D}) = (\boldsymbol{A} \cdot \boldsymbol{C}) \cdot (\boldsymbol{B} \cdot \boldsymbol{D}) - (\boldsymbol{A} \cdot \boldsymbol{D}) \cdot (\boldsymbol{B} \cdot \boldsymbol{C}) \tag{10.1}$$

(d)

$$(\boldsymbol{A} \times \boldsymbol{B}) \times (\boldsymbol{C} \times \boldsymbol{D}) = (\boldsymbol{ABD})\boldsymbol{C} - (\boldsymbol{ABC})\boldsymbol{D}, \tag{10.2}$$

where the symbol $(\boldsymbol{XYZ})$ means the triple scalar product of the three vectors.

10.1 Solution Ten

1. Recall that the cross product can be written as follows:

$$(\boldsymbol{A} \times \boldsymbol{B})_k = A_i B_j \epsilon_{ijk} \tag{10.3}$$

and the dot product can be written as

$$\boldsymbol{A} \cdot \boldsymbol{B} = A_i B_j \delta_{ij} \tag{10.4}$$

Therefore

$$(\boldsymbol{A} \times \boldsymbol{B}) \cdot (\boldsymbol{C} \times \boldsymbol{D}) = A_i B_j \epsilon_{ijk} C_m D_n \epsilon_{mnl} \delta_{kl} \tag{10.5}$$

The tensor structure is

$$\epsilon_{ijk}\epsilon_{mnl}\delta_{kl} = \epsilon_{ijk}\epsilon_{mnk} = \epsilon_{kij}\epsilon_{kmn} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm} \tag{10.6}$$

Therefore,

$$\begin{aligned}(\boldsymbol{A} \times \boldsymbol{B}) \cdot (\boldsymbol{C} \times \boldsymbol{D}) &= A_i B_j C_m D_n \left(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}\right) &(10.7)\\ &= (A_i C_i)(B_j D_j) - (A_i D_i)(B_j C_j) &(10.8)\\ &= (\boldsymbol{A} \cdot \boldsymbol{C})(\boldsymbol{B} \cdot \boldsymbol{D}) - (\boldsymbol{A} \cdot \boldsymbol{D})(\boldsymbol{B} \cdot \boldsymbol{C}) &(10.9)\end{aligned}$$

(d) Let's right this expression in tensor notation

$$[(\boldsymbol{A} \times \boldsymbol{B}) \times (\boldsymbol{C} \times \boldsymbol{D})]_r = A_i B_j \epsilon_{ijk} C_m D_n \epsilon_{mnl} \epsilon_{klr} \tag{10.10}$$

The tensor stucture is thus:

$$\epsilon_{ijk}\epsilon_{mnl}\epsilon_{klr} = -\epsilon_{ijk}\epsilon_{lmn}\epsilon_{lkr} = \epsilon_{ijk}\left(\delta_{mk}\delta_{nr} - \delta_{mr}\delta_{nk}\right) = \epsilon_{ijm}\delta_{nr} - \epsilon_{ijn}\delta_{mr} \tag{10.11}$$

Replacing the vector components, we find

$$\begin{aligned}[(\boldsymbol{A} \times \boldsymbol{B}) \times (\boldsymbol{C} \times \boldsymbol{D})]_r &= A_i B_j C_m D_n \left(\epsilon_{ijm}\delta_{nr} - \epsilon_{ijn}\delta_{mr}\right) &(10.12)\\ &= A_i B_j C_m \epsilon_{ijm} D_r - A_i B_j D_n \epsilon_{ijn} C_r &(10.13)\\ &= (\boldsymbol{A} \times \boldsymbol{B})_m C_m D_r - (\boldsymbol{A} \times \boldsymbol{B})_n D_n C_r &(10.14)\end{aligned}$$

Hence

$$(\boldsymbol{A} \times \boldsymbol{B}) \times (\boldsymbol{C} \times \boldsymbol{D}) = [(\boldsymbol{A} \times \boldsymbol{B}) \cdot \boldsymbol{C}]\, \boldsymbol{D} - [(\boldsymbol{A} \times \boldsymbol{B}) \cdot \boldsymbol{D}]\, \boldsymbol{C} \tag{10.15}$$