PHYS 116B
Homework Two
Solutions

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1 Problem One

[Boas, Ch.10, Sec.6, Problem 5]

Write the tensor transformation equations for $\epsilon_{ijk}\epsilon_{mnp}$ to show that this is a (rank 6) tensor (not a pseudotensor). Hint: Write (6.1) for each ϵ and multiply them, being careful not to re-use a pair of summation indices.

1.1 Solution One

6.1 from Boas reads:

$$\epsilon'_{i_1j_1k_1} = \det(A)a_{i_1i_2}a_{j_1j_2}a_{k_1k_2}\epsilon_{i_2j_2k_2} \tag{1.1}$$

where $\det(A) = \pm 1$ for a general orthogonal tranformation. Using this, we find that

$$\epsilon'_{i_1j_1k_1}\epsilon'_{m_1n_1p_1} = \det(A)^2 a_{i_1i_2}a_{j_1j_2}a_{k_1k_2}\epsilon_{i_2j_2k_2}a_{n_1n_2}a_{m_1m_2}a_{p_1p_2}\epsilon_{m_2n_2p_2} \tag{1.2}$$

Since $\det(A) = \pm 1, \det(A)^2 = 1$. Hence, we discover that

$$\epsilon'_{i_1j_1k_1}\epsilon'_{m_1n_1p_1} = a_{i_1i_2}a_{j_1j_2}a_{k_1k_2}a_{n_1n_2}a_{m_1m_2}a_{p_1p_2}\epsilon_{i_2j_2k_2}\epsilon_{m_2n_2p_2} \tag{1.3}$$

which implics that $\epsilon_{ijk}\epsilon_{mnp}$ transforms as a rank-6 tensor.

2 Problem Two

[Boas, Ch.10, Sec.6, Problem 7]

Write the transformation equations for the triple scalar product $\boldsymbol{W} \cdot (\boldsymbol{U} \times \boldsymbol{V})$ remembering that now $\det \boldsymbol{A} = -1$ if the transformation involves a reflection. Thus show that the triple scalar product of three polar vectors is a pseudoscalar as claimed in Example 2. Hint: Use the result in (6.3).

2.1 Solution Two

Let $\boldsymbol{U}, \boldsymbol{V}$ and $\boldsymbol{W}$ be vectors. Then, consider the quantity $\boldsymbol{W} \cdot (\boldsymbol{U} \times \boldsymbol{V})$. This can be writen as

$$\boldsymbol{W} \cdot (\boldsymbol{U} \times \boldsymbol{V}) = \delta_{ij} W_i (\boldsymbol{U} \times \boldsymbol{V})_j = \delta_{ij} \epsilon_{mnj} W_a U_m V_n \tag{2.1}$$

Acting on this tensor with an orthogonal transformation A (noting that $\delta'_{ij} = \delta_{ij}$ and $\epsilon'_{mnj} = \det(A) a_{mr} a_{ns} a_{jt} \epsilon_{rst}$), we find

$$\delta'_{\alpha\beta} \epsilon'_{\mu\nu\beta} W'_\alpha U'_\mu V'_\nu = \delta_{\alpha\beta} \left(\det(A) a_{\mu m} a_{\nu n} a_{\beta j} \epsilon_{mnj}\right) \left(a_{\alpha i} W_i\right) \left(a_{\mu s} U_s\right) \left(a_{\nu t} V_t\right) \tag{2.2}$$

Using

$$a_{ij} a_{ik} = (A^T)_{ji} (A^T)_{ik} = (A^T A)_{ik} = \delta_{ik} \tag{2.3}$$

we find

$$\begin{aligned}
\delta'_{\alpha\beta} \epsilon'_{\mu\nu\beta} W'_\alpha U'_\mu V'_\nu &= \left(\delta_{\alpha\beta} a_{\alpha i} a_{\beta j}\right) \left(a_{\mu s} a_{\mu m}\right) \left(a_{\nu t} a_{\nu n}\right) \det(A) \epsilon_{mnj} W_i U_s V_t && (2.4) \\
&= \delta_{ij} \delta_{sm} \delta_{tn} \det(A) \epsilon_{mnj} W_i U_s V_t && (2.5) \\
&= \det(A) \delta_{ij} W_i \epsilon_{mnj} U_m V_n && (2.6) \\
&= \det(A) \boldsymbol{W} \cdot (\boldsymbol{U} \times \boldsymbol{V}) && (2.7)
\end{aligned}$$

Therefore, under reflections, i.e. transformation such that $\det(A) = -1$, we find that $\boldsymbol{W} \cdot (\boldsymbol{U} \times \boldsymbol{V}) \to -\boldsymbol{W} \cdot (\boldsymbol{U} \times \boldsymbol{V})$. Therefore, $\boldsymbol{W} \cdot (\boldsymbol{U} \times \boldsymbol{V})$ is a psuedo-scalar.

3 Problem Three

[Boas, Ch.10, Sec.6, Problem 11]

In the following physics formula:

$$\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F}, \tag{3.1}$$

identify each symbol as a vector (polar vector) or a pseudovector (axial vector). Use results from the text and the fact that both sides of an equation must have the same tensor character. The definition of the symbols used is: r = displacement, t = time, m = mass, q = electric charge, v = velocity, F = force, ω = angular velocity, τ = torque, L = angular momentum, T = kinetic energy, E = electric field, B = magnetic field. Assume that t, m, and q are scalars. Note that we are working in 3 dimensional physical space and assuming classical (that is nonrelativistic) physics.

3.1 Solution Three

The quanities $\boldsymbol{r}$ and $\boldsymbol{F}$, the positon and force, are vectors. Do to the cross product, the quantiy $\boldsymbol{r} \times \boldsymbol{F}$ is a pseudo-vector. Therefore, torque, $\boldsymbol{\tau}$, is a pseudo-vector.

4 Problem Four

[Boas, Ch.10, Sec.6, Problem 13]

In the following physics formula:

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \boldsymbol{E}, \tag{4.1}$$

identify each symbol as a vector (polar vector) or a pseudovector (axial vector). Use results from the text and the fact that both sides of an equation must have the same tensor character. The definition of the symbols used is: r = displacement, t = time, m = mass, q = electric charge, v = velocity, F = force, ω = angular velocity, τ = torque, L = angular momentum, T = kinetic energy, E = electric field, B = magnetic field. Assume that t, m, and q are scalars. Note that we are working in 3 dimensional physical space and assuming classical (that is nonrelativistic) physics.

4.1 Solution Four

The electric field $\boldsymbol{E}$ is a vector. From the previous homework, we showed that $\nabla_i = \frac{\partial}{\partial x_i}$ is also a vector (see solutions to problem four of homework 1 with u removed.) Therefore, $\partial \boldsymbol{B}/\partial t$ is a pseudo-vector and hence, $\boldsymbol{B}$ is a pseudo-vector.

5 Problem Five

[Boas, Ch.10, Sec.8, Problem 1]

Find ds^2 in spherical coordinates by the method used to obtain (8.5) for cylindrical coordinates. Use your result to find for spherical coordinates, the scale factors, the vector $d\boldsymbol{s}$, the volume element, the basis vectors $\boldsymbol{a}_r$, $\boldsymbol{a}_\theta$, $\boldsymbol{a}_\phi$ and the corresponding unit basis vectors $\boldsymbol{e}_r$, $\boldsymbol{a}_\theta$, $\boldsymbol{e}_\phi$. Write the g_{ij} matrix.

5.1 Solution Five

Recall that rectangular coordinates (x, y, z) are related to spherical coordinates ρ, θ, ϕ by

$$x = \rho \sin\theta \cos\phi \tag{5.1}$$
$$y = \rho \sin\theta \sin\phi \tag{5.2}$$
$$z = \rho \cos\theta \tag{5.3}$$

Taking differentials of both sides of the above equations, one finds

$$dx = \sin\theta\cos\phi d\rho + \rho\cos\theta\cos\phi d\theta - \rho\sin\theta\sin\phi d\phi \tag{5.4}$$
$$dy = \sin\theta\sin\phi d\rho + \rho\cos\theta\sin\phi d\theta + \rho\sin\theta\cos\phi d\phi \tag{5.5}$$
$$dz = \cos\theta d\rho - \rho\sin\theta d\theta \tag{5.6}$$

Squaring and summing these terms, we find

$$ds^2 = dx^2 + dy^2 + dz^2 = d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{5.7}$$

The scale factors are therefore

$$h_\rho = 1 \tag{5.8}$$
$$h_\theta = \rho \tag{5.9}$$
$$h_\phi = \rho\sin\theta \tag{5.10}$$

The vector $d\boldsymbol{s}$ is

$$d\boldsymbol{s} = \boldsymbol{e}_r d\rho + \boldsymbol{e}_\theta \rho d\theta + \boldsymbol{e}_\phi \rho\sin\theta d\phi \tag{5.11}$$

In terms of $\hat{\imath}, \hat{\jmath}$ and $\hat{k}$, the unit vector are

$$\boldsymbol{e}_\rho = \hat{\imath}\sin\theta\cos(\phi) + \hat{\jmath}\sin\theta\sin\phi + \hat{k}\cos\theta \tag{5.12}$$
$$\boldsymbol{e}_\theta = \hat{\imath}\cos\theta\cos\phi + \hat{\jmath}\cos\theta\sin\phi - \hat{k}\sin\theta \tag{5.13}$$
$$\boldsymbol{e}_\phi = \hat{\jmath}\cos\phi - \hat{\imath}\sin\phi \tag{5.14}$$

The vectors $\boldsymbol{a}_\rho, \boldsymbol{a}_\theta$ and $\boldsymbol{a}_\phi$ are given by

$$\boldsymbol{a}_\rho = \boldsymbol{e}_\rho \tag{5.15}$$
$$\boldsymbol{a}_\theta = \rho \boldsymbol{a}_\theta \tag{5.16}$$
$$\boldsymbol{a}_\phi = \rho \sin\theta \boldsymbol{a}_\phi \tag{5.17}$$

Lastly, the metrix tensor is given by

$$g_{ij} = \boldsymbol{a}_i \boldsymbol{a}_j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & \rho^2 \sin^2\theta \end{pmatrix} \tag{5.18}$$

6 Problem Six

[Boas, Ch.10, Sec.8, Problem 2]

Observe that a simpler way to find the velocity $d\boldsymbol{s}/dt$ in (8.10) is to divide the vector $d\boldsymbol{s}$ in (8.6) by dt. Complete the problem to find the acceleration in cylindrical coordinates.

6.1 Solution Six

In cylindrical coordinates, $d\boldsymbol{s}$ is given by

$$d\boldsymbol{s} = \boldsymbol{e}_r dr + \boldsymbol{e}_\theta r d\theta + \boldsymbol{e}_z dz \tag{6.1}$$

Dividing this by dt we find

$$\frac{d\boldsymbol{s}}{dt} = \boldsymbol{e}_r \frac{dr}{dt} + \boldsymbol{e}_\theta r \frac{d\theta}{dt} + \boldsymbol{e}_z \frac{dz}{dt} = \boldsymbol{e}_r \dot{r} + \boldsymbol{e}_\theta r\dot{\theta} + \boldsymbol{e}_z \dot{z} \tag{6.2}$$

which is (8.10). To get the acceleration, we should take the time derivative of the above expression. To do so, we use

$$\frac{d}{dt}\boldsymbol{e}_r = \boldsymbol{e}_\theta \dot{\theta} \tag{6.3}$$

$$\frac{d}{dt}\boldsymbol{e}_\theta = -\boldsymbol{e}_r \dot{\theta} \tag{6.4}$$

Thus,

$$\frac{d^2\boldsymbol{s}}{dt} = \boldsymbol{e}_r \ddot{r} + \boldsymbol{e}_\theta \dot{r}\dot{\theta} + \boldsymbol{e}_\theta(\dot{r}\dot{\theta} + r\ddot{\theta}) - \boldsymbol{e}_r r\dot{\theta}^2 + \boldsymbol{e}_z \ddot{z} = \boldsymbol{e}_r \left(\ddot{r} - r\dot{\theta}^2\right) + \boldsymbol{e}_\theta \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right) + \boldsymbol{e}_z \ddot{z} \tag{6.5}$$

7 Problem Seven

[Boas, Ch.10, Sec.8, Problem 5]

Using the results of Problem five, express the vector in Boas-10.8.4 in spherical coordinates.

7.1 Solution Seven

We would like to express the vector

$$\boldsymbol{V} = y\boldsymbol{i} - x\boldsymbol{j} + \boldsymbol{k} \tag{7.1}$$

in terms of spherical coordinates. Part of this is very simple. We know that

$$x = \rho \sin\theta \cos\phi \tag{7.2}$$
$$y = \rho \sin\theta \sin\phi \tag{7.3}$$
$$z = \rho \cos\theta \tag{7.4}$$

Next, we need to determine $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ in terms of $\boldsymbol{e}_\rho, \boldsymbol{e}_\theta, \boldsymbol{e}_\phi$. To do this, recall form problem five that

$$\boldsymbol{e}_\rho = \hat{\imath}\sin\theta\cos\phi + \hat{\jmath}\sin\theta\sin\phi + \hat{k}\cos\theta \tag{7.5}$$
$$\boldsymbol{e}_\theta = \hat{\imath}\cos\theta\cos\phi + \hat{\jmath}\cos\theta\sin\phi - \hat{k}\sin\theta \tag{7.6}$$
$$\boldsymbol{e}_\phi = \hat{\jmath}\cos\phi - \hat{\imath}\sin\phi \tag{7.7}$$

We can write this as a matrix equation:

$$\begin{pmatrix} \boldsymbol{e}_\rho \\ \boldsymbol{e}_\theta \\ \boldsymbol{e}_\phi \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix} \begin{pmatrix} \hat{\imath} \\ \hat{\jmath} \\ \hat{k} \end{pmatrix} \tag{7.8}$$

Inverting the matrix, we find

$$\begin{pmatrix} \hat{\imath} \\ \hat{\jmath} \\ \hat{k} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{e}_\rho \\ \boldsymbol{e}_\theta \\ \boldsymbol{e}_\phi \end{pmatrix} \tag{7.9}$$

or

$$\hat{\imath} = \sin\theta\cos\phi\boldsymbol{e}_\rho + \cos\theta\cos\phi\boldsymbol{e}_\theta - \sin\phi\boldsymbol{e}_\phi \tag{7.10}$$
$$\hat{\jmath} = \sin\theta\sin\phi\boldsymbol{e}_\rho + \cos\theta\sin\phi\boldsymbol{e}_\theta + \cos\phi\boldsymbol{e}_\phi \tag{7.11}$$
$$\hat{k} = \cos\theta\boldsymbol{e}_\rho - \sin\theta\boldsymbol{e}_\theta \tag{7.12}$$

Therefore, we find that

$$\begin{aligned} \boldsymbol{V} &= \rho \sin\theta \sin\phi (\sin\theta\cos\phi \boldsymbol{e}_\rho + \cos\theta\cos\phi\boldsymbol{e}_\theta - \sin\phi\boldsymbol{e}_\phi) \\ &\quad - \rho\sin\theta\cos\phi(\sin\theta\sin\phi\boldsymbol{e}_\rho + \cos\theta\sin\phi\boldsymbol{e}_\theta + \cos\phi\boldsymbol{e}_\phi) \\ &\quad + (\cos\theta\boldsymbol{e}_\rho - \sin\theta\boldsymbol{e}_\theta) \end{aligned} \tag{7.13}$$

$$= \cos\theta\boldsymbol{e}_\rho - \sin\theta\,(\boldsymbol{e}_\theta + \rho\boldsymbol{e}_\phi) \tag{7.14}$$

8 Problem Eight

[Boas, Ch.10, Sec.8, Problem 7]

As in problem five, find ds^2, the scale factors, the vector $d\boldsymbol{s}$, the volume (or area) element, the $\boldsymbol{a}$ vectors, and the $\boldsymbol{e}$ vectors in elliptic cylinder coordinates u, v, z:

$$x = a\cosh(u)\cos(v), \tag{8.1}$$
$$y = a\sinh(u)\sin(v), \tag{8.2}$$
$$z = z. \tag{8.3}$$

8.1 Solution Eight

The differentials are

$$dx = a\sinh(u)\cos(v)du - a\cosh(u)\sin(v)dv \tag{8.4}$$
$$dy = a\cosh(u)\sin(v)du + a\sinh(u)\cos(v)dv \tag{8.5}$$
$$dz = dz \tag{8.6}$$

Using these, we fidn that

$$ds^2 = dz^2 + \frac{1}{2}a^2(\cosh(2u) - \cos(2v))\left(du^2 + dv^2\right) \tag{8.7}$$

or, using $\cosh(2u) = \sinh^2(u) + \cosh^2(u), \cosh^2(u) - \sinh^2(u) = 1,\ \cos(2v) = \cos^2(v) - \sin^2(v), \cos^2(v) + \sin^2(v) = 1$

$$ds^2 = dz^2 + a^2(\sinh^2(u) + \sin^2(v))\left(du^2 + dv^2\right) \tag{8.8}$$

The scale factors are therefore

$$h_u = \frac{a}{\sqrt{2}}\sqrt{\cosh(2u) - \cos(2v)} = a\sqrt{\sinh^2(u) + \sin^2(v)} \tag{8.9}$$
$$h_v = \frac{a}{\sqrt{2}}\sqrt{\cosh(2u) - \cos(2v)} = a\sqrt{\sinh^2(u) + \sin^2(v)} \tag{8.10}$$
$$h_z = 1 \tag{8.11}$$

The vector $d\boldsymbol{s}$ is

$$d\boldsymbol{s} = a\sqrt{\sinh^2(u) + \sin^2(v)}\,(\boldsymbol{e}_u du + \boldsymbol{e}_v dv) + \boldsymbol{e}_z dz \tag{8.12}$$

The vectors $\boldsymbol{e}_u, \boldsymbol{e}_v, \boldsymbol{e}_z$ are

$$\boldsymbol{e}_u = \frac{\boldsymbol{i}\sinh(u)\cos(v) + \boldsymbol{j}\cosh(u)\sin(v)}{\sqrt{\sinh^2(u) + \sin^2(v)}} \tag{8.13}$$
$$\boldsymbol{e}_v = \frac{\boldsymbol{j}\sinh(u)\cos(v) - \boldsymbol{i}\cosh(u)\sin(v)}{\sqrt{\sinh^2(u) + \sin^2(v)}} \tag{8.14}$$
$$\boldsymbol{e}_z = \boldsymbol{k} \tag{8.15}$$

The vectors $\boldsymbol{a}_u, \boldsymbol{a}_v, \boldsymbol{a}_z$ are

$$\boldsymbol{a}_u = a\sqrt{\sinh^2(u) + \sin^2(v)}\boldsymbol{e}_u \tag{8.16}$$

$$\boldsymbol{a}_v = a\sqrt{\sinh^2(u) + \sin^2(v)}\boldsymbol{e}_v \tag{8.17}$$

$$\boldsymbol{a}_z = \boldsymbol{e}_z \tag{8.18}$$

The volume element is

$$dV = h_u h_v h_z dudvdz = a\left(\sinh^2(u) + \sin^2(v)\right) dudvdz \tag{8.19}$$

Lastly, the metrix tensor is

$$g_{ij} = \boldsymbol{a}_i \boldsymbol{a}_j = \begin{pmatrix} a^2\left(\sinh^2(u) + \sin^2(v)\right) & 0 & 0 \\ 0 & a^2\left(\sinh^2(u) + \sin^2(v)\right) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{8.20}$$

9 Problem Nine

[Boas, Ch.10, Sec.8, Problem 8]

As in problem five, find ds^2, the scale factors, the vector $d\boldsymbol{s}$, the volume (or area) element, the $\boldsymbol{a}$ vectors, and the $\boldsymbol{e}$ vectors in parabolic cylinder coordinates u, v, ϕ:

$$x = uv\cos\phi, \tag{9.1}$$
$$y = uv\sin\phi, \tag{9.2}$$
$$z = \frac{1}{2}(u^2 - v^2). \tag{9.3}$$

9.1 Solution Nine

The differentials are

$$dx = \cos\phi\,(vdu + udv) - uv\sin\phi d\phi \tag{9.4}$$
$$dy = \sin\phi\,(vdu + udv) + uv\cos\phi d\phi \tag{9.5}$$
$$dz = udu - vdv \tag{9.6}$$

Using these, we fidn that

$$ds^2 = \left(u^2 + v^2\right)\left(du^2 + dv^2\right) + u^2v^2d\phi^2 \tag{9.7}$$

The scale factors are therefore

$$h_u = \sqrt{u^2 + v^2} \tag{9.8}$$
$$h_v = \sqrt{u^2 + v^2} \tag{9.9}$$
$$h_\phi = uv \tag{9.10}$$

The vector $d\boldsymbol{s}$ is

$$d\boldsymbol{s} = \sqrt{u^2 + v^2}\,(\boldsymbol{e}_u du + \boldsymbol{e}_v dv) + uv\boldsymbol{e}_\phi d\phi \tag{9.11}$$

The vectors $\boldsymbol{e}_u, \boldsymbol{e}_v, \boldsymbol{e}_z$ are

$$\boldsymbol{e}_u = \frac{\boldsymbol{i}v\cos\phi + \boldsymbol{j}v\sin\phi + \boldsymbol{k}u}{\sqrt{u^2 + v^2}} \tag{9.12}$$
$$\boldsymbol{e}_v = \frac{\boldsymbol{i}u\cos\phi + \boldsymbol{j}u\sin\phi - \boldsymbol{k}v}{\sqrt{u^2 + v^2}} \tag{9.13}$$
$$\boldsymbol{e}_\phi = \boldsymbol{j}\cos\phi - \boldsymbol{i}\sin\phi \tag{9.14}$$

The vectors $\boldsymbol{a}_u, \boldsymbol{a}_v, \boldsymbol{a}_z$ are

$$\boldsymbol{a}_u = \sqrt{u^2 + v^2}\boldsymbol{e}_u \tag{9.15}$$
$$\boldsymbol{a}_v = \sqrt{u^2 + v^2}\boldsymbol{e}_v \tag{9.16}$$
$$\boldsymbol{a}_\phi = uv\boldsymbol{e}_\phi \tag{9.17}$$

The volume element is

$$dV = h_u h_v h_\phi du dv d\phi = uv(u^2 + v^2) du dv d\phi \tag{9.18}$$

Lastly, the metrix tensor is

$$g_{ij} = \boldsymbol{a}_i \boldsymbol{a}_j = \begin{pmatrix} u^2 + v^2 & 0 & 0 \\ 0 & u^2 + v^2 & 0 \\ 0 & 0 & u^2 v^2 \end{pmatrix} \tag{9.19}$$

10 Problem Ten

[Boas, Ch.10, Sec.8, Problem 12]

Using the expression you have found for $d\boldsymbol{s}$, and for the $\boldsymbol{e}$ vectors, find the velocity and acceleration components in elliptic cylinder coordinates.

10.1 Solution Ten

In problem 8, we found $d\boldsymbol{s}$ is elliptical coordinates. Dividing the $d\boldsymbol{s}$ we found by dt, we find that the velocity in elliptical coordinates is

$$\frac{d\boldsymbol{s}}{dt} = a\kappa\left(\boldsymbol{e}_u\dot{u} + \boldsymbol{e}_v\dot{v}\right) + \boldsymbol{e}_z\dot{z} \tag{10.1}$$

where we've defined

$$\kappa(u,v) = \sqrt{\sinh^2(u) + \sin^2(v)} \tag{10.2}$$

The acceleration is given by

$$\frac{d\boldsymbol{s}}{dt} = a\kappa(u,v)\left(\frac{d\boldsymbol{e}_u}{dt}\ddot{u} + \boldsymbol{e}_u\dot{u} + \frac{d\boldsymbol{e}_v}{dt}\dot{v} + \boldsymbol{e}_v\ddot{v}\right) \tag{10.3}$$

$$+ a\frac{d\kappa}{dt}\left(\boldsymbol{e}_u\dot{u} + \boldsymbol{e}_v\dot{v}\right) + \boldsymbol{e}_z\ddot{z} \tag{10.4}$$

In order to compute $d^2\boldsymbol{s}/dt^2$, we need $d\boldsymbol{e}_u/dt$, $d\boldsymbol{e}_v/dt$ and the time derivative of $\kappa(u,v)$. These are

$$\frac{d\boldsymbol{e}_u}{dt} = \frac{1}{2\kappa^2}f(u,v)\boldsymbol{e}_v \tag{10.5}$$

$$\frac{d\boldsymbol{e}_u}{dt} = -\frac{1}{2\kappa^2}f(u,v)\boldsymbol{e}_u \tag{10.6}$$

$$\frac{d\kappa}{dt} = \frac{1}{\kappa}g(u,v) \tag{10.7}$$

where

$$f = \dot{v}\sinh(2u) - \dot{u}\sin(2v) \tag{10.8}$$

$$g = \dot{u}\sinh(2u) + \dot{v}\sin(2u) \tag{10.9}$$

Hence, we find that

$$\frac{d^2\boldsymbol{s}}{dt^2} = \frac{a}{2\kappa}\left(2\kappa^2\ddot{u} - f\dot{v} + 2g\dot{u}\right)\boldsymbol{e}_u + \frac{a}{2\kappa}\left(2\kappa^2\ddot{v} + f\dot{u} + 2g\dot{v}\right)\boldsymbol{e}_v + \boldsymbol{e}_z\ddot{z} \tag{10.10}$$