PHYS 116C
Homework Three

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1 Problem One

[Boas, Ch.8, Sec.1, Problem 5]

Find the position x of a particle at time t if its acceleration is $d^2x/dt^2 = A\sin(\omega t)$.

1.1 Solution One

Integrating $x''(t)$ onces, we obtain

$$\frac{dx}{dt} = \int \frac{d^2x}{dt^2}dt = -\frac{A}{\omega}\cos(\omega t) + \mathcal{C} \tag{1.1}$$

where $\mathcal{C}$ is an integration constant. Integrating once again, we find

$$x(t) = \int \frac{dx}{dt}dt = -\frac{A}{\omega^2}\sin(\omega t) + \mathcal{C}t + \mathcal{D} \tag{1.2}$$

where $\mathcal{D}$ is a second integration constant.

2 Problem Two

[Boas, Ch.8, Sec.1, Problem 4]

Find the distance which an object moves in time t if it starts from rest and has an acceleration $d^2x/dt^2 = ge^{-kt}$. Show that for small t the result is approximately (1.10), and for very large t, the speed dx/dt is approximately constant. The constant is called the terminal speed. (This problem corresponds roughly to the motion of a parachutist.)

2.1 Solution Two

Integrating the acceleration, we find

$$\frac{dx}{dt} = \int \frac{d^2x}{dt^2} dt = -\frac{g}{k}e^{-kt} + \mathcal{C} \tag{2.1}$$

Given that the object was released from rest, i.e. $dx/dt(t = 0) = 0$, we find that $\mathcal{C} = g/k$. Integrating once more, we find

$$x(t) = \int \frac{dx}{dt} dt = \frac{g}{k^2}e^{-kt} + \frac{g}{k}t + \mathcal{D} \tag{2.2}$$

Calling $x(0) = x_0$, we can see that $\mathcal{D} = x_0 - g/k^2$. Therefore, our solution is

$$x(t) = x_0 + \frac{g}{k^2}\left(e^{-kt} - 1 + kt\right) \tag{2.3}$$

For small t, we can write the exponential as

$$e^{-kt} = 1 - kt + \frac{1}{2}k^2t^2 + \mathcal{O}(t^3) \tag{2.4}$$

Thus, for small t, we can write

$$x(t) = x_0 + \frac{1}{2}gt^2 + \mathcal{O}(t^3) \tag{2.5}$$

which is of the form of eqn. 1.10 from Boas chapter 10. The velocity as a functions of time is

$$v(t) = -\frac{g}{k}e^{-kt} + \frac{g}{k} \tag{2.6}$$

As $t \to \infty$, we find that $v \to g/k$. Thus, g/k is the terminal velocity.

3 Problem Three

[Boas, Ch.8, Sec.2, Problem 7]

For the following differential equation:

$$ydy + (xy^2 - 8x)dx = 0, \qquad y = 3 \text{ when } x = 1 \tag{3.1}$$

separate variables and find a solution containing one arbitrary constant. Then find the value of the constant to give a particular solution satisfying the given boundary condition. Computer plot a slope field and some of the solution curves.

3.1 Solution Three

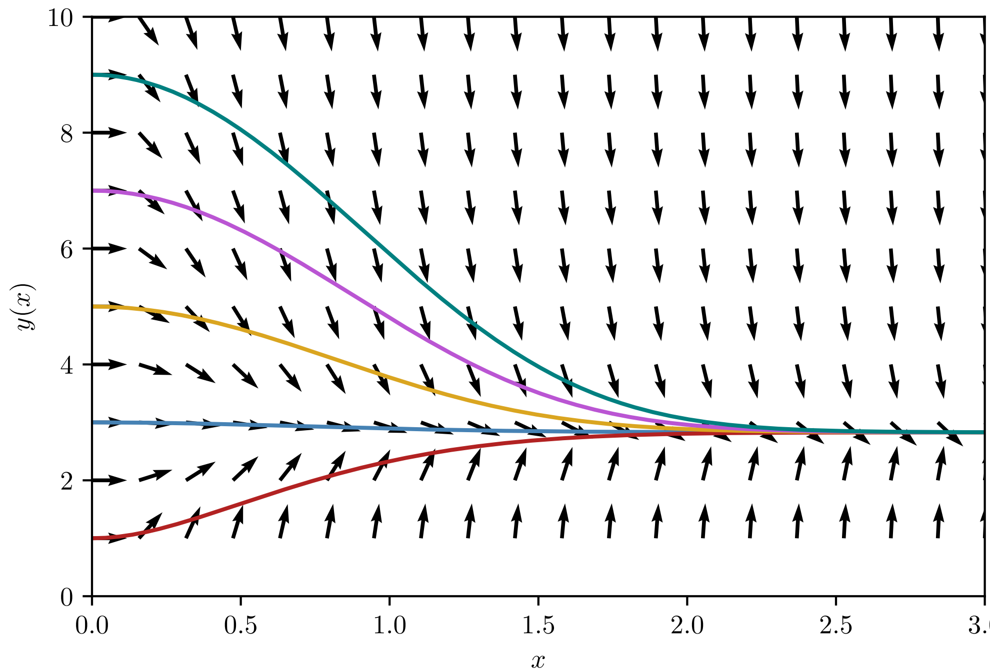

Figure 1: Family of curves for the differential equation $y' = x(8 - y^2)/y$

The differential equation can be separated to the form

$$\frac{y}{8 - y^2}dy = xdx \tag{3.2}$$

Integrating both sides of this equation, we find

$$-\frac{1}{2}\log\left(8 - y^2\right) = \frac{1}{2}x^2 + \mathcal{C} \tag{3.3}$$

where $\mathcal{C}$ is an arbitrary constant. Solving for y, we find

$$y = \pm\sqrt{8 + \mathcal{D}e^{-x^2}} \tag{3.4}$$

where $\mathcal{D}$ is some other integration constant. Given that $y = 3$ at $x = 1$, i.e. $y > 0$, we can see that the $-$ solution can be dropped. Additionally, we can see that at $x = 1$, the argument of the square root must be 9. Hence, $\mathcal{D} = e$. Hence, our solution is

$$y(x) = \sqrt{8 + e^{1-x^2}} \tag{3.5}$$

4 Problem Four

[Boas, Ch.8, Sec.2, Problem 11]

For the following differential equation:

$$2y' = 3(y-2)^{1/3}, \qquad y = 3 \;\text{ when }\; x = 1 \tag{4.1}$$

separate variables and find a solution containing one arbitrary constant. Then find the value of the constant to give a particular solution satisfying the given boundary condition. Computer plot a slope field and some of the solution curves.

4.1 Solution Four

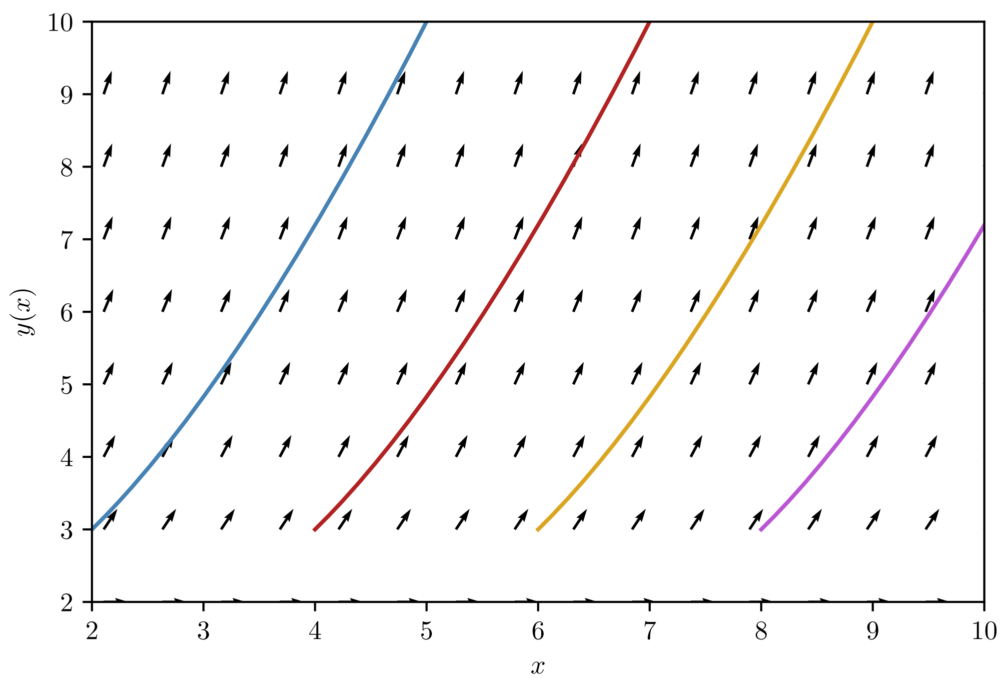

Figure 2: Family of curves for the differential equation $y' = 3(y-2)^{1/3}/2$

This differential equation is

$$\frac{dy}{(y-2)^{1/3}}dy = \frac{3}{2}dx \tag{4.2}$$

Integrating, we find

$$\frac{3}{2}(y-2)^{2/3} = \frac{3}{2}x + \mathcal{C} \tag{4.3}$$

or

$$y = 2 \pm \sqrt{(x + \mathcal{C})^3} \tag{4.4}$$

Given the initial condition, the $-$ solution doesn't work. Using $y(x = 1) = 3$, we find that $\mathcal{C} = 0$. Thus

$$y(x) = 2 + x^{3/2} \tag{4.5}$$

5 Problem Five

[Boas, Ch.8, Sec.3, Problem 3]

Using (3.9), find the general solution of each of the following differential equation:

$$dy + (2xy - xe^{-x^2})dx = 0. \tag{5.1}$$

Compare a computer solution and, if necessary, reconcile it with yours. Hint: See comments just after (3.9), and Example 1.

5.1 Solution Five

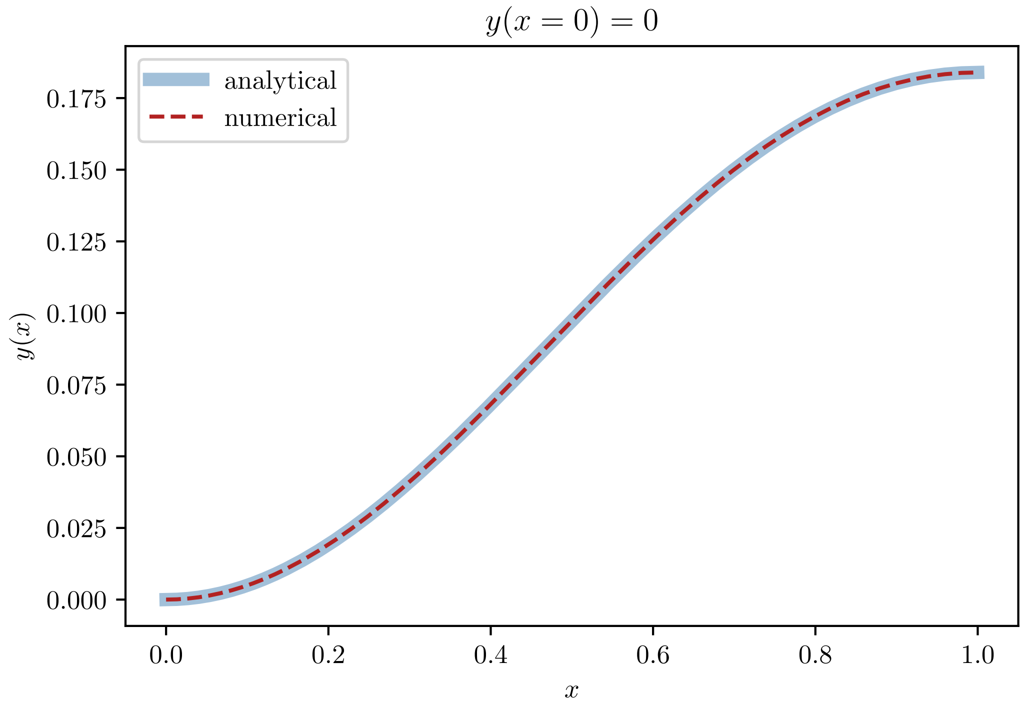

Figure 3: Analytic vs. numerical solution for the differential equation $y' = -2xy + xe^{-x^2}$

We can rewrite this differential equation as

$$\frac{dy}{dx} + 2xy = xe^{-x^2} \tag{5.2}$$

To solve this, we use the method of integrating factor. Our integrating factor will be

$$I(x) = e^{\int 2x dx} = e^{x^2} \tag{5.3}$$

Multiplying through by this factor, we find

$$e^{x^2}\frac{dy}{dx} + 2xe^{x^2}y = x \tag{5.4}$$

We notice that the left-hand-side is just the total derivative of ye^{x^2}. Therefore

$$\frac{d}{dx}\left(e^{x^2}y(x)\right) = x \tag{5.5}$$

Integrating both sides and multiplying by e^{-x^2}, we find that

$$y(x) = \left(\frac{1}{2}x^2 + \mathcal{C}\right)e^{-x^2} \tag{5.6}$$

6 Problem Six

[Boas, Ch.8, Sec.3, Problem 10]

Using (3.9), find the general solution of each of the following differential equation:

$$y' + y\tanh(x) = 2e^x. \tag{6.1}$$

Compare a computer solution and, if necessary, reconcile it with yours. Hint: See comments just after (3.9), and Example 1.

6.1 Solution Six

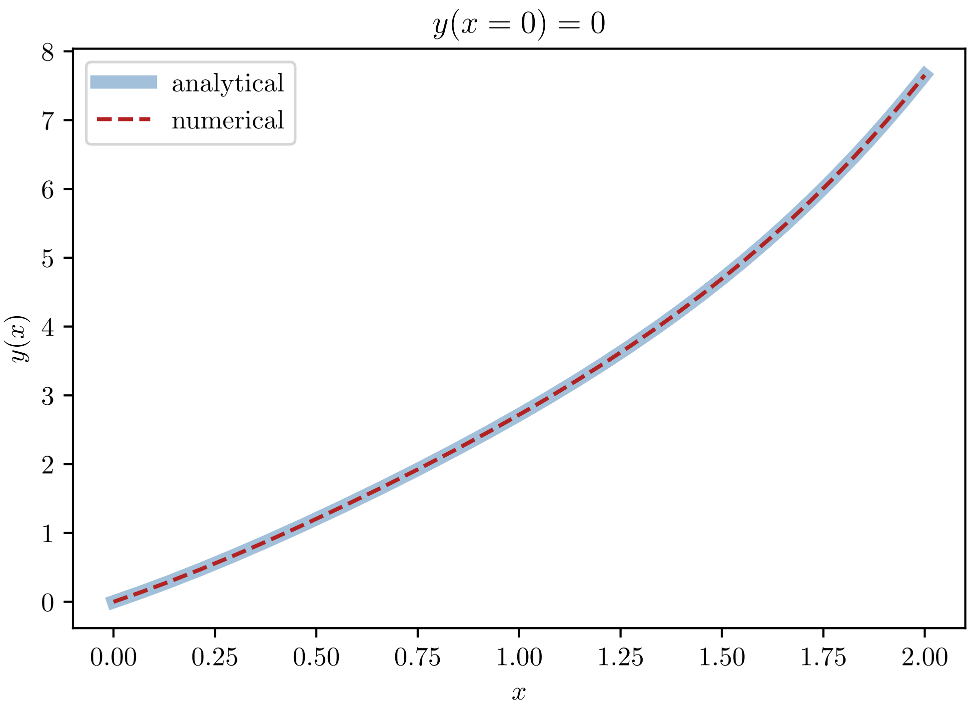

Figure 4: Analytic vs. numerical solution for the differential equation $y' = -y\tanh(x) + 2e^x$

To solve this differential equation, we again use an integrating factor. This time, our integrating factor is

$$I(x) = e^{\int \tanh(x)dx} = e^{\ln(\cosh(x))} = \cosh(x) \tag{6.2}$$

Multiplying through by this integrating factor, our differential equation can be written as

$$\cosh(x)\frac{dy}{dx} + \sinh(x)y(x) = 2\cosh(x)e^x \tag{6.3}$$

or

$$\frac{d}{dx}\left(\cosh(x)y(x)\right) = e^{2x} + 1 \tag{6.4}$$

where we used $\cosh(x) = (e^x + e^{-x})/2$ on the right-hand-side. Integrating both sides, we find

$$\cosh(x)y(x) = \frac{1}{2}e^{2x} + x + \mathcal{C} \tag{6.5}$$

which we can write as

$$y(x) = \frac{e^{2x} + 2x + \mathcal{D}}{2\cosh(x)} \tag{6.6}$$

7 Problem Seven

[Boas, Ch.8, Sec.4, Problem 4]

Use the methods of Boas section 10.4 to solve the following differential equation:

$$(2xe^{3y} + e^x)dx + (3x^2e^{3y} - y^2)dy = 0. \tag{7.1}$$

Compare computer solutions and reconcile differences.

7.1 Solution Seven

We recognize this differential equation as an exact differential equation. We can see this by setting

$$P(x, y) = 2xe^{3y} + e^x \tag{7.2}$$
$$Q(x, y) = 3x^2e^{3y} - y^2 \tag{7.3}$$

Differentiating these we find that

$$\frac{dP(x, y)}{dy} = 6xe^{3y} \tag{7.4}$$
$$\frac{dQ(x, y)}{dx} = 6xe^{3y} \tag{7.5}$$
$$\tag{7.6}$$

Hence, this differential equation is exact. We can then write

$$P(x, y) = \frac{\partial F}{\partial x} \tag{7.7}$$
$$Q(x, y) = \frac{\partial F}{\partial y} \tag{7.8}$$

Then, our differential equation reads $dF = 0$, which implies that $F(x, y)$ is a constant. Integrating $\partial F/\partial x$, we find

$$F(x, y) = f(y) + \int \frac{\partial F}{\partial x}dx = f(y) + x^2e^{3y} + e^x \tag{7.9}$$

where $f(y)$ is an unknown function of y. Integrating $\partial F/\partial y$, we find

$$F(x, y) = g(y) + \int \frac{\partial F}{\partial y}dx = g(y) + x^2e^{3y} - \frac{1}{3}y^3 \tag{7.10}$$

Matching these two expressions, we find that

$$F(x, y) = x^2e^{3y} - \frac{1}{3}y^3 + e^x \tag{7.11}$$

We can then solve for $y(x)$ by using $F(x, y) = \text{constant} = \mathcal{C}$

$$x^2e^{3y} - \frac{1}{3}y^3 + e^x = \mathcal{C} \tag{7.12}$$

8 Problem Eight

[Boas, Ch.8, Sec.4, Problem 7]

Use the methods of Boas section 10.4 to solve the following differential equation:

$$x^2 dy + (y^2 - xy)dx = 0 \tag{8.1}$$

Compare computer solutions and reconcile differences.

8.1 Solution Eight

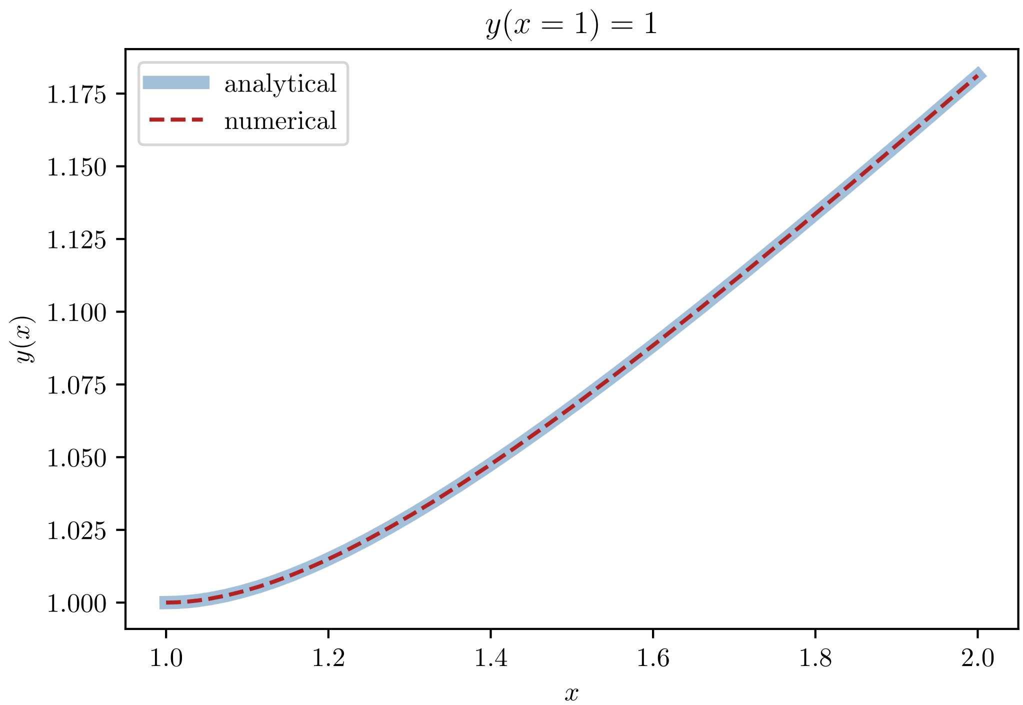

Figure 5: Analytic vs. numerical solution for the differential equation $y' = \frac{xy - y^2}{x^2} = \frac{y}{x} - \left(\frac{y}{x}\right)^2$

Our differential equation reads

$$\frac{dy}{dx} = \frac{xy - y^2}{x^2} = \frac{y}{x} - \left(\frac{y}{x}\right)^2 \tag{8.2}$$

We notice that the right-hand-side is a function of y/x only. Thus, we set $v(x) = y(x)/x$. Then, we find a new differential equation in $v(x)$ given by

$$x\frac{dv}{dx} + v = v - v^2 \tag{8.3}$$

We can solve this differntial equation by separation of variable. The differential equation separates to

$$\frac{dv}{v^2} = -\frac{1}{x} \tag{8.4}$$

Integrating both sides, we find

$$-\frac{1}{v} = -\log(x) + \mathcal{C} \tag{8.5}$$

which gives us

$$v = \frac{1}{\log(x) + \mathcal{D}} \tag{8.6}$$

and therefore

$$y(x) = \frac{x}{\log(x) + \mathcal{D}} \tag{8.7}$$

9 Problem Nine

[Boas, Ch.8, Sec.4, Problem 8]

Use the methods of Boas section 10.4 to solve the following differential equation:

$$ydy = (-x + \sqrt{x^2 + y^2})dx \tag{9.1}$$

Compare computer solutions and reconcile differences.

9.1 Solution Nine

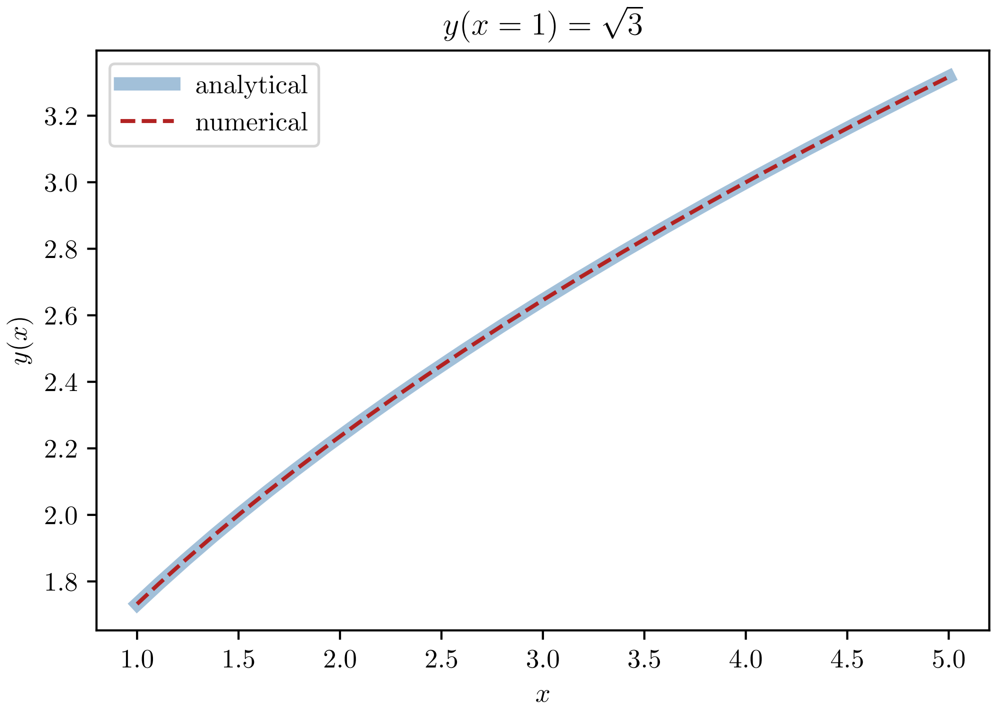

Figure 6: Analytic vs. numerical solution for the differential equation $y' = \frac{xy - y^2}{x^2} = -\frac{x}{y} + \sqrt{1 + \left(\frac{x}{y}\right)^2}$

Our differential equation reads

$$\frac{dy}{dx} = -\frac{x}{y} + \sqrt{1 + \left(\frac{x}{y}\right)^2} \tag{9.2}$$

We can see that the right-hand-side is a function of y/x only. Thus, we set $v(x) = y/x$ and obtain the following differential equation for v:

$$x\frac{dv}{dx} + v = -\frac{1}{v} + \sqrt{1 + 1/v^2} \tag{9.3}$$

Multiplying through by v and moving all v's to the right-hand-side, we find

$$xv\frac{dv}{dx} = -\left(1 + v^2\right) + \sqrt{1 + v^2} \tag{9.4}$$

We can now separate, findingvariablesvariables

$$\frac{vdv}{(1 + v^2) - \sqrt{1 + v^2}} = -dx/x \tag{9.5}$$

To integrate the left-hand-side, we make a change of variables $u = \sqrt{1 + v^2}$, with $du = vdv/\sqrt{1 + v^2}$. Then

$$\int \frac{vdv}{(1 + v^2) - \sqrt{1 + v^2}} = \int \frac{1}{u - 1} = \log(u - 1) = \log\left(\sqrt{1 + v^2} - 1\right) \tag{9.6}$$

Therefore, we find that

$$\log\left(\sqrt{1 + v^2} - 1\right) = -\log(x) + \mathcal{C} \tag{9.7}$$

We can then untangle this to obtain y:

$$y(x) = \pm x\sqrt{\left(1 + \mathcal{C}/x\right)^2 - 1} \tag{9.8}$$

10 Problem Ten

[Boas, Ch.8, Sec.4, Problem 13]

Use the methods of Boas section 10.4 to solve the following differential equation:

$$yy' - 2y^2 \cot(x) = \sin(x)\cos(x) \tag{10.1}$$

Compare computer solutions and reconcile differences.

10.1 Solution Ten

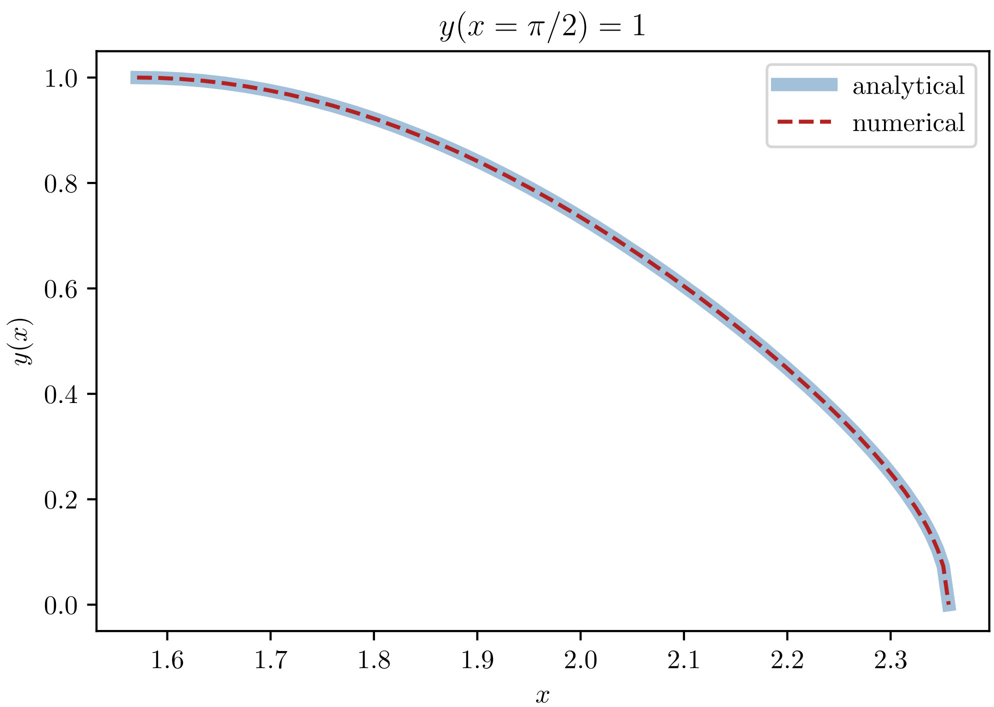

Figure 7: Analytic vs. numerical solution for the differential equation $\frac{dy}{dx} - 2y\cot(x) = \sin(x)\cos(x)/y$

If we divide our differential equation by y, we can see that it is of the form of Bernoulli's equation with $n = -1$. We thus make the change of variables $z = y^2$ with $z' = 2yy'$. Then, our differential equation for z is

$$\frac{1}{2}\frac{dz}{dx} - 2\cot(x)z = \sin(x)\cos(x) \tag{10.2}$$

We can solve this by using integrating factor, with

$$I(x) = e^{-4\int \cot(x)dx} = e^{-4\log(\sin(x))} = \sin^{-4}(x) \tag{10.3}$$

Multiplying through by $I(x)$, we find

$$\sin^{-4}(x)\frac{dz}{dx} - 4\frac{\cos(x)}{\sin^5(x)}z = 2\frac{\cos(x)}{\sin^3(x)} \tag{10.4}$$

which we can write as

$$\frac{d}{dx}\left(\frac{z(x)}{\sin^4(x)}\right) = 2\frac{\cos(x)}{\sin^3(x)} \tag{10.5}$$

Integrating both sides, we find

$$z(x) = \sin^4(x)\left(-\cot^2(x) + \mathcal{C}\right) = \mathcal{C}\sin^4(x) - \sin^2(x)\cos^2(x) \tag{10.6}$$

and hence,

$$y(x) = \pm\sqrt{\mathcal{C}\sin^4(x) - \sin(2x)^2/4} \tag{10.7}$$