PHYS 116C
Homework Four

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1 Problem One

[**Boas, Ch.8, Sec.4, Problem 14**] Use the methods of Ch. 8, Sec. 4 of Boas to solve the following differential equation:

$$(x-1)y' + y - x^{-2} + 2x^{-3} = 0. \tag{1.1}$$

Compare computer solutions and reconcile differences.

1.1 Solution One

First, we rewrite the differential equation as follows:

$$(x-1)y' + y = x^{-2} - 2x^{-3}. \tag{1.2}$$

We notice that the left-hand-side is just a total derivative:

$$\frac{d}{dx}\left((x-1)y\right) = (x-1)y' + y \tag{1.3}$$

We can thus simply integrate both sides. Integrating the right-hand-side, we find

$$\int dx \left(\frac{1}{x^2} - \frac{2}{x^3}\right) = -\frac{1}{x} + \frac{1}{x^2} + \mathcal{C} \tag{1.4}$$

Integrating the left-hand-side we get $(x-1)y$. Therefore,

$$(x-1)y = -\frac{1}{x} + \frac{1}{x^2} + \mathcal{C} = -\left(\frac{x-1}{x^2}\right) + \mathcal{C} \tag{1.5}$$

Thus, our solution is

$$y(x) = -\frac{1}{x^2} + \frac{\mathcal{C}}{x-1} \tag{1.6}$$

Suppose that $y(2) = n$. Then

$$y(2) = n = -\frac{1}{4} + \mathcal{C} \qquad \Longrightarrow \qquad \mathcal{C} = \frac{4n+1}{4} \tag{1.7}$$

Then

$$y(x) = -\frac{1}{x^2} + \frac{4n+1}{4(x-1)} \tag{1.8}$$

Fig. (1) shows the numerical and analytic solutions to the differential equation for various values of n.

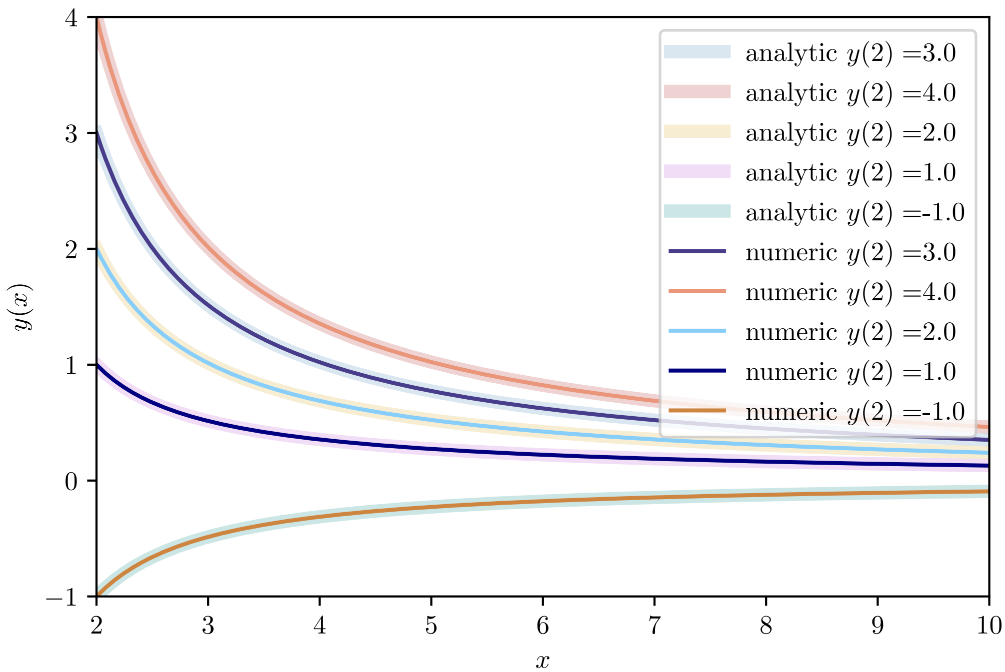

Figure 1: Numerical vs. analytic solutions to the differential equation $(x-1)y' + y = x^{-2} - 2x^{-3}$. The analytic solutions are show with large line widths and are faded while the numerical are bold with thin linewidth.

2 Problem Two

[Boas, Ch.8, Sec.4, Problem 19] Find the family of curves satisfying the differential equation

$$(x+y)dy + (x-y)dx = 0 \tag{2.1}$$

and also find their orthogonal trajectories.

2.1 Solution Two

First let's find the solutions to the differential equation. Let $P(x,y) = x+y$ and $Q(x,y) = x-y$. We notice that P and Q are homogenous functions of degree 1. That is

$$P(x,y) = x\left(1 + y/x\right) \tag{2.2}$$

$$Q(x,y) = x\left(1 - y/x\right) \tag{2.3}$$

Thus, our differential equation is homoegeneous. We can write our differential equation as

$$\frac{dy}{dx} = -\frac{Q}{P} = -\left(\frac{1-y/x}{1+y/x}\right) \tag{2.4}$$

To solve this equation, we make the following change of variables: $y = xv$. Then, $y' = xv' + v$. We then have the following differnetial equation for v:

$$xv' + v = -\frac{1-v}{1+v} \tag{2.5}$$

We can rewrite this equation by moving the v on the LHS to the RHS, yeilding

$$xv' = -\left(v + \frac{1-v}{1+v}\right) = -\frac{1+v^2}{1+v} \tag{2.6}$$

We thus can seperate the differential equation:

$$\frac{1+v}{1+v^2}dv = -\frac{dx}{x} \tag{2.7}$$

The integral of the LHS is

$$\int \frac{1+v}{1+v^2}dv = \int \frac{1}{1+v^2}dv + \int \frac{v}{1+v^2}dv = \arctan(v) + \frac{1}{2}\log\left(1+v^2\right) \tag{2.8}$$

Therefore, we find that

$$\arctan(v) + \frac{1}{2}\log\left(1+v^2\right) = -\log(x) + \mathcal{C} \tag{2.9}$$

This equation is begging for us to use $v = \tan(\theta)$. For $v = \tan(\theta)$, we can set $y = r\sin\theta$ and $x = r\cos\theta$, i.e. polar coordinates. Plugging these in, we find

$$\theta - \log(\cos\theta) = -\log(r\cos\theta) + \mathcal{C} \tag{2.10}$$

Using $\log(ab) = \log(a) + \log(b)$, we find

$$\theta = -\log(r) + \mathcal{C} \tag{2.11}$$

which is solved by $r(\theta) = \mathcal{C}'e^{-\theta}$. These solutions are decaying spirals. The solid curves in Fig. (2) show the solutions for $\mathcal{C}' = 0.5, 1.0, 1.5, 2.0$.

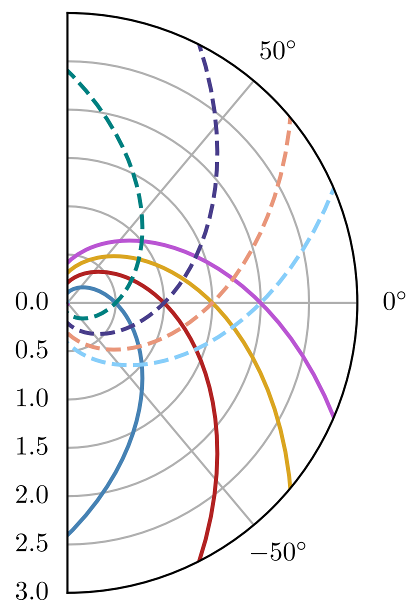

Figure 2: Solutions and orthogonal solutions to $(x + y)dy + (x - y)dx = 0$. Solid lines are the solutions and the dashed lines are the orthogonal solutions.

Let's now solve for the orthogonal solutions. The orthogonal differential equation is obtained by negating and inverting the slope. That is, we change the slope to

$$y' \to -\frac{1}{y'} \tag{2.12}$$

Then, our differential equation is

$$-\frac{1}{y'} = -\left(\frac{1 - y/x}{1 + y/x}\right) \tag{2.13}$$

which can be written as

$$y' = \frac{1 + y/x}{1 - y/x} \tag{2.14}$$

We again have a homogeneous equation. We follow the above steps to obtain

$$xv' + v = \frac{1 + v}{1 - v} \tag{2.15}$$

This differential equation can be rearranged to obtain

$$xv' = \frac{v^2 + 1}{1 - v} \tag{2.16}$$

This can now be separated out:

$$\frac{1 - v}{1 + v^2} dv = \frac{dx}{x} \tag{2.17}$$

Integrating the entire equation, we find

$$\arctan(v) - \frac{1}{2} \log\left(1 + v^2\right) = \log(x) + \mathcal{C} \tag{2.18}$$

Changing to polar coordinates again, we find

$$\theta + \log(\cos\theta) = \log(r\cos\theta) + \mathcal{C} \tag{2.19}$$

Solving for r yeilds $r = \mathcal{C}' e^{\theta}$. The dashed curves in Fig. (2) show the solutions for $\mathcal{C}' = 0.5, 1.0, 1.5, 2.0$. We can see that the dashed lines are perpendicular to the solid lines everywhere.

3 Problem Three

[**Boas, Ch.8, Sec.4, Problem 21**] As in text just before (4.11) of Boas Ch. 8, show that

(a) $x^2 - 5xy + y^3/x$ is a homogeneous function of degree 2;

(b) $x^{-1}(y^4 - x^3y) - xy^2\sin(x/y)$ is homogeneous of degree 3;

(c) $x^2y^3 + x^5\ln(y/x) - y^6/\sqrt{x^2+y^2}$ is homogeneous of degree 5;

(d) $x^2 + y, x + \cos(y)$ and $y + 1$ are not homogeneous.

See Boas Ch. 4, Sec. 13. Problem 1 for a more general definition of a homogeneous function of any number of variables.

3.1 Solution Three

(a) If $x^2 - 5xy + y^3/x$ is a homogeneous function of degree 2, then we should be able to write it as $x^2 f(y/x)$. Indeed, if we factor our an x^2, we obtain

$$x^2 - 5xy + y^3/x = x^2\left(1 - 5y/x + (y/x)^3\right) \tag{3.1}$$

(b) If $x^{-1}(y^4 - x^3y) - xy^2\sin(x/y)$ is a homogeneous function of degree 3, then we should be able to write it as $x^3 f(y/x)$. Indeed, if we factor our an x^3, we obtain

$$x^{-1}(y^4 - x^3y) - xy^2\sin(x/y) = x^3\left((y/x)^4 - (y/x) - (y/x)^2\sin(1/(y/x))\right) \tag{3.2}$$

(c) If $x^2y^3 + x^5\ln(y/x) - y^6/\sqrt{x^2+y^2}$ is a homogeneous function of degree 5, then we should be able to write it as $x^5 f(y/x)$. Indeed, if we factor our an x^3, we obtain

$$x^2y^3 + x^5\ln(y/x) - y^6/\sqrt{x^2+y^2} = x^5\left((y/x)^3 + \ln(y/x) - (y/x)^5/\sqrt{1+(y/x)^2}\right) \tag{3.3}$$

(d) Let's try to write $x^2 + y$ as a homogeneous function. We would want it to be of degree 2. Trying to put it in the correct for gives us

$$x^2 + y = x^2\left(1 + \frac{1}{x}(y/x)\right) \tag{3.4}$$

which is not homogeneous. We can see that $x + \cos(y)$ is not homogeneous right away since it containes $\cos(y)$. This factor screws everything up. You can see this by taylor expanding the cosine:

$$\cos(y) = 1 - \frac{1}{2}y^2 + \frac{1}{4!}y^4 + \cdots \tag{3.5}$$

There is x^n you can multiply $\cos(y)$ by to make it homogeneous. Lastly, $1 + y$ cannot be homogeneous. It has y's and no x's.

4 Problem Four

[Boas, Ch.8, Sec.4, Problem 25] An equation of the form

$$y' = f(x)y^2 + g(x)y + h(x) \tag{4.1}$$

is called a *Riccati* equation. If we know one particular solution y_p, then the substitution $y = y_p + 1/z$ gives a linear first-order equation for z. We can solve this for z and substitute back to find a solution of the y equation containing one arbitrary constant (see Problem 26). Following this method, check the given y_p, and then solve

(a) $y' = xy^2 - \frac{2}{x}y - \frac{1}{x^3}, \qquad y_p = \frac{1}{x^2};$

(b) $y' = \frac{2}{x}y^2 + \frac{1}{x}y - 2x, \qquad y_p = x;$

(c) $y' = e^{-x}y^2 + y - e^x, \qquad y_p = e^x;$

4.1 Solution Four

(a) First let's verify the solution give is indeed a solution. if $y_p = 1/x^2$, then $y_p' = -2/x^3$. The LHS is

$$xy_p^2 - \frac{2}{x}y_p - \frac{1}{x^3} = \frac{x}{x^4} - \frac{2}{x^3} - \frac{1}{x^3} = -\frac{2}{x^3} \tag{4.2}$$

Thus, y_p is indeed a solution. Now we will use $y = y_p + 1/z$. Nearly the entire differential equation is linear. The non-linear piece comes from the y^2. This give

$$y^2 = y_p^2 + 2y_p\frac{1}{z} + \frac{1}{z^2} \tag{4.3}$$

Plugging this in, and using our knowledge of the fact that y_p is a solution, we find

$$-\frac{z'}{z^2} = x\left(\frac{2}{x}\frac{1}{z} + \frac{x}{z^2}\right) - \frac{2}{x}\frac{1}{z} \tag{4.4}$$

Simplifying this, we arrive at $z' = -x$. Hence, $z = -\frac{1}{2}x^2 + \frac{1}{2}\mathcal{C}$ (I put in the 1/2 on the $\mathcal{C}$ for convienience.). Our general solutions is therefore

$$y(x) = \frac{1}{x^2} + \frac{2}{\mathcal{C} - x^2} = \frac{\mathcal{C} + x^2}{x^2(\mathcal{C} - x^2)} \tag{4.5}$$

(b) First, let's check that $y_p = x$ is a solution. The LHS is simply 1. The RHS is

$$\frac{2}{x}y_p^2 + \frac{1}{x}y_p - 2x = \frac{2}{x}x^2 + \frac{1}{x}x - 2x = 1 \tag{4.6}$$

Thus, $y_p = x$ is indeed a solution. Bearing this in mind and setting $y = y_p + 1/z$, we find

$$-\frac{z'}{z^2} = \frac{2}{x}\left(\frac{2x}{z} + \frac{1}{z^2}\right) + \frac{1}{xz} = \frac{1}{z}\left(4 + \frac{1}{x} + \frac{2}{xz}\right) \tag{4.7}$$

Mulitplying through by z^2, we find

$$z' + \left(4 + \frac{1}{x}\right) z = -\frac{2}{x} \tag{4.8}$$

This equation can be solved by integrating factor. Our integrating factor will be

$$I(x) = \exp\left(\int \left(4 + \frac{1}{x}\right) dx\right) = \exp(4x + \log(x)) = xe^{4x} \tag{4.9}$$

Multipkying through by this factor, we find

$$\frac{d}{dx}\left(zxe^{4x}\right) = -2e^{4x} \tag{4.10}$$

Integrating both sides, we find

$$z = -\frac{1}{2x}\left(1 - \mathcal{C}e^{-4x}\right) \tag{4.11}$$

Thus, our entire solution is

$$y(x) = x - \frac{2x}{1 - \mathcal{C}e^{-4x}} = -x\left(\frac{1 + \mathcal{C}e^{-4x}}{1 - \mathcal{C}e^{-4x}}\right) \tag{4.12}$$

(c) First we check that $y_p = e^x$ is a solution. Of cource $y_p' = y_p$. The LHS is

$$e^{-x}y^2 + y - e^x = e^x + e^x - e^x = e^x \tag{4.13}$$

Thus, y_p is a solution. Next, we plug in $y = y_p + 1/z$:

$$-\frac{z'}{z^2} = e^{-x}\left(\frac{2}{z}e^x + \frac{1}{z^2}\right) + \frac{1}{z} = \frac{3}{z} + \frac{e^{-x}}{z^2} \tag{4.14}$$

Rearranging, we find

$$z' + 3z = -e^{-x} \tag{4.15}$$

We can solve this by use of an integrating factor $I = e^{3x}$. Using this, we find

$$\frac{d}{dx}\left(ze^{3x}\right) = -e^{2x} \tag{4.16}$$

After integrating, we find

$$z = -\frac{1}{2}e^{-x} + \frac{1}{2}\mathcal{C}e^{-3x} = -\frac{e^{-x}}{2}\left(1 - \mathcal{C}e^{-2x}\right) \tag{4.17}$$

Thus, our solution is

$$y(x) = e^x - \frac{2e^x}{1 - \mathcal{C}e^{-2x}} = -e^x\left(\frac{1 + \mathcal{C}e^{-2x}}{1 - \mathcal{C}e^{-2x}}\right) \tag{4.18}$$

5 Problem Five

[**Boas, Ch.8, Sec.5, Problem 7**] Solve the following differential equation by the methods discussed in Boas Ch. 8, Sec. 5 and compare computer solutions.

$$(D^2 - 5D + 6)y = 0 \tag{5.1}$$

5.1 Solution Five

To solve this differential equation, we guess $y = Ae^{\omega x}$. Plugging this into the differential equation, we find

$$Ae^{\omega x}\left(\omega^2 - 5\omega + 6\right) = 0 \tag{5.2}$$

Thus, in order to have non-trivial solutions (i.e. $A = 0$), we require that $\omega^2 - 5\omega + 6 = 0$. Hence, $\omega = 2, 3$. Thus, our soultion is

$$y(x) = c_1 e^{2x} + c_2 e^{3x} \tag{5.3}$$

6 Problem Six

[**Boas, Ch.8, Sec.5, Problem 10**] Solve the following differential equation by the methods discussed in Boas Ch. 8, Sec. 5 and compare computer solutions.

$$y'' - 2y' = 0 \tag{6.1}$$

6.1 Solution Six

To solve this differential equation, we guess $y(x) = Ae^{\omega x}$. Plugging this in, we find

$$Ae^{\omega x}\left(\omega^2 - 2\omega\right) = 0 \tag{6.2}$$

Thus, either $\omega = 0$ or $\omega = 2$. Our general solution is therefore

$$y(x) = c_1 + c_2 e^{2x} \tag{6.3}$$

7 Problem Seven

[**Boas, Ch.8, Sec.5, Problem 12**] Solve the following differential equation by the methods discussed in Boas Ch. 8, Sec. 5 and compare computer solutions.

$$(2D^2 + D - 1)y = 0 \tag{7.1}$$

7.1 Solution Seven

As in the previous two problems, we guess $y = Ae^{\omega x}$. Plugging this in. we find

$$Ae^{\omega x}\left(2\omega^2 + \omega - 1\right) = 0 \tag{7.2}$$

For non-trivial solutions, we require $2\omega^2 + \omega - 1 = 0$. That is, we require that

$$\omega = \frac{-1 \pm 3}{4} = -1, \frac{1}{2} \tag{7.3}$$

Thus, our general solution is

$$y(x) = c_1 e^{-x} + c_2 e^{x/2} \tag{7.4}$$

8 Problem Eight

[Boas, Ch.8, Sec.5, Problem 13] Recall from Boas Ch. 3, equation (8.5), that a set of functions is linearly independent if their Wronskian is not identically zero. Calculate the Wronskian of each of the set

$$e^{-x}, e^{-4x} \tag{8.1}$$

to show that they are linearly independent. Write the differential equation of which they are solutions. Also note that each set of functions is a set of basis functions for a linear vector space (see Boas Ch. 3, Sec. 14, Example 2) and that the general solution of the differential equation gives all vectors of the vector space.

8.1 Solution Eight

The Wronskian for the functions e^{-x} and e^{-4x} is

$$W = \begin{vmatrix} e^{-x} & e^{-4x} \\ -e^{-x} & -4e^{-4x} \end{vmatrix} = -4e^{-5x} + e^{-5x} = -3e^{-5x} \neq 0 \tag{8.2}$$

Since the Wronskian doesn't vanish, we know these functions are linearly independent. The differential equation that these functions satisfy is

$$(D+1)(D+4)y = 0 \tag{8.3}$$

9 Problem Nine

[**Boas, Ch.8, Sec.5, Problem 27**] Use the results of Problem 21 to find the general solutions of the following equation:

$$D^2(D-1)^2(D+2)^3y = 0 \tag{9.1}$$

and compare computer solutions.

9.1 Solution Nine

We solve this differential equation by guessing $y = Ae^{\omega x}$. Plugging this in, we find

$$Ae^{\omega x}\omega^2(\omega-1)^2(\omega+2)^3 = 0 \tag{9.2}$$

Thus, we require that, either $\omega = 0, 1$ or -2. However, our differential equation is 7^{th} order, meaning we expect 7 linearly independent solutions. To find the remaining, we use the standard trick when we have repeated roots. That is, we guess a polynomial times the know solutions. We guess x, xe^x, xe^{-2x} and x^2e^{-2x}. We can see that x will work due to the D^2. Let's check xe^x:

$$(D-1)^2xe^x = (D-1)\left(e^x + xe^x - xe^x\right) = (D-1)e^x = 0 \tag{9.3}$$

Thus, xe^x works. Next, let's check xe^{-2x}:

$$(D+2)^3xe^{-2x} = (D+2)^2\left(e^{-2x} - 2xe^{-2x} + 2e^{-2x}\right) = (D+2)^2e^{-2x} = 0 \tag{9.4}$$

Lastly, let's check x^2e^{-2x}:

$$\begin{aligned}
(D+2)^3x^2e^{-2x} &= (D+2)^2\left(2xe^{-2x} - 2x^2e^{-2x} + 2x^2e^{-2x}\right) && (9.5)\\
&= (D+2)^2 2xe^{-2x} && (9.6)\\
&= 2(D+2)\left(e^{-2x} - 2xe^{-2x} + 2xe^{-2x}\right) && (9.7)\\
&= 2(D+2)e^{-2x} && (9.8)\\
&= 0 && (9.9)
\end{aligned}$$

Thus, we have found all of our solutions. One can also show that these functions are indeed linearly independent. The Wronskian is $93312e^{-4x}$. Our general solutions is therefore,

$$y = c_1 + c_2x + c_3e^x + c_4xe^x + c_5e^{-2x} + c_6xe^{-2x} + c_7x^2e^{-2x} \tag{9.10}$$

10 Problem Ten

[Boas, Ch.8, Sec.5, Problem 38] Solve the RLC circuit equation [(5.33) or (5.34) of Boas Ch. 8] with $V = 0$ as we did (5.27), and write the conditions and solutions for overdamped, critically damped, and under- damped electrical oscillations in terms of the quantities R, L, and C.

10.1 Solution Ten

The RLC circuit equation for the current is

$$LI'' + RI' + I/C = 0 \tag{10.1}$$

So solve this equation, we guess $I = Ae^{\omega t}$. Plugging this in, we find

$$Ae^{\omega t}\left(L\omega^2 + R\omega + \frac{1}{C}\right) = 0 \tag{10.2}$$

The solutions are

$$\omega = \frac{-R \pm \sqrt{R^2 - 4L/C}}{2L} = \frac{R}{2L}\left(-1 \pm \sqrt{1 - 4L/R^2C}\right) \tag{10.3}$$

Let $\gamma = (R/2L)\sqrt{1 - 4L/R^2C}$. If $\gamma \neq 0$, then we have the following solutions:

$$I(t) = c_1 e^{-Rt/2L+\gamma t} + c_2 e^{-Rt/2L-\gamma t} \tag{10.4}$$

If γ is real, then our solution is overdamped. That is, if

$$CR^2 > 4L \qquad \Longrightarrow \qquad \text{overdamped} \tag{10.5}$$

The overdamped solutions are

$$I(t) = e^{-Rt/2L}\left(c_1 e^{\gamma t} + c_2 e^{-\gamma t}\right) \tag{10.6}$$

If γ is imaginary, our solution is underdamped:

$$CR^2 < 4L \qquad \Longrightarrow \qquad \text{underdamped} \tag{10.7}$$

The underdamped solutions are

$$I(t) = e^{-Rt/2L}\left(c_1 e^{i|\gamma|t} + c_2 e^{-i|\gamma|t}\right) \tag{10.8}$$

If $\gamma = 0$, or $R^2C = 4L$, then our solution is critically damped. In this case, our solutions are

$$I(t) = e^{-Rt/2L}\left(c_1 + c_2 t\right) \tag{10.9}$$