PHYS 116C
Homework Five

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1 Problem One

[Boas, Ch.8, Sec.6, Problem 9]

Find the general solution of the following differential equations (complementary function + particular solution):

$$(D^2 + 2D + 1)y = 2e^{-x} \tag{1.1}$$

Find the particular solution by inspection or by (6.18), (6.23), or (6.24). Also find a computer solution and reconcile differences if necessary, noticing especially whether the particular solution is in simplest form [see (6.26) and the discussion after (6.15)].

1.1 Solution One

Let's first find the complementary solution. We will guess $y = e^{\alpha x}$. With this guess we find

$$(D^2 + 2D + 1)e^{\alpha x} = e^{\alpha x}(\alpha^2 + 2\alpha + 1) = e^{\alpha x}(\alpha + 1)^2 = 0 \tag{1.2}$$

We can see that for $e^{\alpha x}$ to be a solution, $\alpha = -1$. Since this is a double root, we know that the solution is

$$y_c = (A + Bx)e^{-x} \tag{1.3}$$

For the particular solution, we will guess $x^2 e^{-x}$:

$$(D^2 + 2D + 1)x^2 e^{-x} = 2e^{-x} \tag{1.4}$$

Therefore, we can se that the particular solution is $y_p = x^2 e^{-x}$. Thus, the entire solution is

$$y = y_c + y_p = (A + Bx)e^{-x} + x^2 e^{-x} \tag{1.5}$$

2 Problem Two

[Boas, Ch.8, Sec.6, Problem 18]

Find the general solution of the following differential equations (complementary function + particular solution):

$$(D^2 + 2D + 17)y = 60e^{-4x}\sin(5x) \tag{2.1}$$

Find the particular solution by inspection or by (6.18), (6.23), or (6.24). Also find a computer solution and reconcile differences if necessary, noticing especially whether the particular solution is in simplest form [see (6.26) and the discussion after (6.15)].

2.1 Solution Two

First, let's determine the complementary solution. As usual we guess $e^{\alpha x}$:

$$(D^2 + 2D + 17)e^{\alpha x} = e^{\alpha x}(\alpha^2 + 2\alpha + 17) = 0 \tag{2.2}$$

The solution to this is $\alpha = -1 \pm 4i$. Thus, the complementary solution is

$$y_c(x) = c_1 e^{(-1-4i)x} + c_1 e^{(-1+4i)x} = e^{-x}\left(c_1 e^{-4ix} + c_2 e^{4ix}\right) \tag{2.3}$$

For the particular solution, we will guess $e^{-4x}\cos(5x)$. After some algebra, one finds that:

$$(D^2 + 2D + 17)e^{-4x}\cos(5x) = 30e^{-4x}\sin(5x) \tag{2.4}$$

Which is what we want up to a factor of two. Thus, our entire solution is

$$y(x) = e^{-x}\left(c_1 e^{-4ix} + c_2 e^{4ix}\right) + 2e^{-4x}\cos(5x) \tag{2.5}$$

The real solutions can be obtained by replacing

$$c_1 e^{-4ix} + c_2 e^{4ix} = (c_1 + c_2)\cos(4x) + i(c_1 - c_2)\sin(x) = c_1'\cos(4x) + c_2'\sin(x), \tag{2.6}$$

resulting in

$$y(x) = e^{-x}\left(c_1'\cos(4x) + c_2'\sin(x)\right) + 2e^{-4x}\cos(5x) \tag{2.7}$$

3 Problem Three

[Boas, Ch.8, Sec.6, Problem 27] 423.27

Verify that (6.4) is a particular solution of (6.2). Verify that another particular solution of (6.2) is

$$y_p = \frac{1}{10}\sin(2x) - e^{-x} \tag{3.1}$$

Observe that we obtain the same general solution (6.7) whichever particular solution we use [since $(A-1)$ is just as good an arbitrary constant as A]. Show in general that the difference between two particular solutions of $(a_2D^2 + a_1D + a_0)y = f(x)$ is always a solution of the homogeneous equation $a_2D^2 + a_1D + a_0)y = 0$, and thus show that the general solution is the same for all choices of a particular solution.

3.1 Solution Three

Equation (6.2) is

$$(D^2 + 5D + 4)y = \cos(2x) \tag{3.2}$$

First let's verify that $\sin(2x)/10$ is a solution. Notice that

$$D\sin(2x) = 2\cos(2x) \tag{3.3}$$
$$D^2\sin(2x) = -4\sin(2x) \tag{3.4}$$

Using these, we find that

$$(D^2 + 5D + 4)\sin(2x) = -4\sin(2x) + 10\cos(2x) + 4\sin(2x) = 10\cos(2x) \tag{3.5}$$

Thus, $(D^2+5D+4)\sin(2x)/10 = \cos(2x)$. Next, let's verify that e^{-x} is also a solution:

$$(D^2 + 5D + 4)e^{-x} = e^{-x} - 5e^{-x} + 4e^{-x} = 0 \tag{3.6}$$

Since $D^2 + 5D + 4$ is linear, we conclude that $\frac{1}{10}\sin(2x) - e^{-x}$ is a solution (note that is is pretty clear since the complementary solution is $c_1e^{-x} + c_2e^{-4x}$.) Suppose we have two particular solutions:

$$(a_2D^2 + a_1D + a_0)y_p^1 = f(x) \tag{3.7}$$
$$(a_2D^2 + a_1D + a_0)y_p^2 = f(x) \tag{3.8}$$

Subtracting the two, we find

$$(a_2D^2 + a_1D + a_0)(y_p^1 - y_p^2) = 0 \tag{3.9}$$

Hence, $y_1 - y_2$ is a solution to the homogeneous differential equation. Therefore we can write two solutions:

$$y_1 = y_p^1 + y_c \tag{3.10}$$
$$y_2 = y_p^2 + y_c \tag{3.11}$$

We can write the second solution as

$$y_2 = y_p^1 + (y_p^2 - y_p^1) + y_c \tag{3.12}$$

But since $y_p^2 - y_p^1$ is a solution to the homogeneous differential equation, $(y_p^2 - y_p^1) + y_c$ is the same as y_c but with different arbitrary constants.

4 Problem Four

[Boas, Ch.8, Sec.6, Problem 31] 424.31

(a) Show that

$$D(e^{ax}y) = e^{ax}(D+a)y, \tag{4.1}$$
$$D^2(e^{ax}y) = e^{ax}(D+a)^2y, \tag{4.2}$$

and so onl that is, for any positive integral n,

$$D^n(e^{ax}y) = e^{ax}(D+a)^ny. \tag{4.3}$$

Thus show that if $L[D]$ is any polynomial in the operator D, then

$$L[D](e^{ax}y) = e^{ax}L[D+a]y \tag{4.4}$$

This is called the *exponetial shift*.

(b) Use (a) to show that

$$(D-1)^3(e^xy) = e^xD^3y, \tag{4.5}$$
$$(D^2+D-6)(e^{-3x}y) = e^{-3x}(D^2-5D)y. \tag{4.6}$$

(c) Replace D by $D-a$, to obtain

$$e^{ax}P[D]y = P[D-a]e^{ax}y. \tag{4.7}$$

This is called the *inverse exponential shift*.

4.1 Solution Four

(a) Let's work on the first identity:

$$D(e^{ax}y) = ae^{ax}y + e^{ax}Dy = e^{ax}(Dy+ay) = e^{ax}(D+a)y \tag{4.8}$$

Now the second identity:

$$D^2(e^{ax}y) = D\left[e^{ax}(D+a)y\right] \tag{4.9}$$
$$= ae^{ax}(D+a)y + e^{ax}D(D+a)y \tag{4.10}$$
$$= e^{ax}\left(D+a\right)\left(D+a\right)y \tag{4.11}$$
$$= e^{ax}\left(D+a\right)^2 y \tag{4.12}$$

Let's now prove that $D^n(e^{ax}y) = e^{ax}(D+a)^ny$. We will do this by mathematical induction. First, we know that $D(e^{ax}y) = e^{ax}(D+a)y$. Next, let's assume that

$D^n(e^{ax}y) = e^{ax}(D+a)^n y$ is true. Now consider $D^{n+1}(e^{ax}y)$. Using the assumption that $D^n(e^{ax}y) = e^{ax}(D+a)^n y$ is true, we find that

$$\begin{aligned} D^{n+1}(e^{ax}y) &= DD^n(e^{ax}y) && (4.13)\\ &= De^{ax}(D+a)^n y && (4.14)\\ &= ae^{ax}(D+a)^n y + e^{ax}D(D+a)^n y && (4.15)\\ &= e^{ax}\left[a(D+a)^n y + D(D+a)^n y\right] && (4.16)\\ &= e^{ax}\left(D+a\right)\left(D+a\right)^n y && (4.17)\\ &= e^{ax}(D+a)^{n+1} y && (4.18) \end{aligned}$$

Thus, if $D^n(e^{ax}y) = e^{ax}(D+a)^n y$ is true, we have that $D^{n+1}(e^{ax}y) = e^{ax}(D+a)^{n+1} y$ must be true. Therefore $D^n(e^{ax}y) = e^{ax}(D+a)^n y$ holds for all $n \in \mathbb{N}$. Next let $L[D]$ be a polynomial of D:

$$L[D] = \sum_{n=0}^{N} a_n D^n \tag{4.19}$$

Then consider $L[D](e^{ax}y)$:

$$\begin{aligned} L[D](e^{ax}y) &= \sum_{n=0}^{N} a_n D^n(e^{ax}y) && (4.20)\\ &= \sum_{n=0}^{N} e^{ax} a_n (D+a)^n y && (4.21)\\ &= e^{ax}\sum_{n=0}^{N} a_n (D+a)^n y && (4.22)\\ &= e^{ax} L[D+a] y && (4.23) \end{aligned}$$

(b) Let's work on the first expression $(D-1)^3(e^x y)$. We can write this as $L[D](e^x y)$ with $L[D] = (D-1)^3$. Using the previous results, we find

$$(D-1)^3(e^x y) = L[D](e^x y) = e^x L[D+1] y = e^x D^3 y \tag{4.24}$$

Next, $L[D] = D^2 + D - 6 = (D+3)(D-2)$. Using this,

$$\begin{aligned} (D^2 + D - 6)(e^{-3x}y) &= L[D](e^{-3x}y) && (4.25)\\ &= e^{-3x} L[D-3] y && (4.26)\\ &= e^{-3x}(D-3+3)(D-3-2) y && (4.27)\\ &= e^{-3x}(D^2 - 5D) y && (4.28) \end{aligned}$$

(c) Using $L[D](e^{ax}y) = e^{ax}L[D+a]y$ and replacing $D \to D-a$, we find

$$L[D-a](e^{ax}y) = e^{ax}L[D-a+a]y = e^{ax}L[D]y \tag{4.29}$$

Hence, $e^{ax}L[D]y = L[D-a](e^{ax}y)$.

5 Problem Five

[Boas, Ch.8, Sec.6, Problem 36] 429.36

Solve the following differential equations by using the principle of superposition [see the solution of equation (6.29)]

$$(D^2+1)y = 2\sin(x) + 4x\cos(x) \tag{5.1}$$

5.1 Solution Five

For convinience, let us define $L[D] = D^2+1$. Our first step is to determine the complementary solution. Guessing $y = e^{\alpha x}$, we find

$$L[D]e^{\alpha x} = e^{\alpha x}L[\alpha] = e^{\alpha x}(\alpha^2+1) = 0 \tag{5.2}$$

Thus, $e^{\alpha x}$ is a solution for $\alpha = \pm i$. Therefore the complementary solution is

$$y_c(x) = Ae^{ix} + Be^{-ix} \tag{5.3}$$

Now let's determine the particular solution. First, let's try to get the $2\sin(x)$ part of the particular solution. We will guess a polynomial times $\sin(x)$. We know that $\sin(x)$ alone wont work since $L[D]\sin(x) = 0$. Let's try $Cx\sin(x) + Dx\cos(x)$. After some algebra, we find

$$L[D]\left(Cx\sin(x) + Dx\cos(x)\right) = 2C\cos(x) - 2D\sin(x) \tag{5.4}$$

Thus, we find that we can obtain the $2\sin(x)$ on the right-hand-side if $y = -x\cos(x)$. Next, let's determine what particular solution give $4x\cos(x)$. We will guess a function of the form $(Cx+Dx^2)\sin(x) + (Ex+Fx^2)\cos(x)$. After some work, we find that

$$L[D]\left((Cx+Dx^2)\sin(x) + (Ex+Fx^2)\cos(x)\right) \tag{5.5}$$

$$= 2\left[(C+F+2Dx)\cos(x) + (D-E-2Fx)\sin(x)\right] \tag{5.6}$$

We can immediately see that we need $C = F = 0$. This leaves us with

$$2\left[2Dx\cos(x) + (D-E)\sin(x)\right] \tag{5.7}$$

(note that at this point we can see that we could have taken care of the entire $2\sin(x) + 4x\cos(x)$ with $D = 2, E = 0$.) To remove the $\sin(x)$, we need $D = E$. To get the $4x\cos(x)$, we set $D = 1$. Thus, to get the $4x\cos(x)$, we need $y = x^2\sin(x) + x\cos(x)$. Now, by the superposition principle (or linearity of the differential equation), we have the the particular solution is

$$y_p = -x\cos(x) + x^2\sin(x) + x\cos(x) = x^2\sin(x) \tag{5.8}$$

Therefore, our entire solution is

$$y = x^2 \sin(x) + Ae^{ix} + Be^{-ix} \tag{5.9}$$

Or, if we want a purely real solution (with potentially complex coefficients), we replace

$$\begin{aligned} Ae^{ix} + Be^{-ix} &= A\cos(x) + iA\sin(x) + B\cos(x) - iB\sin(x) && (5.10) \\ &= (A+B)\cos(x) + i(A-B)\sin(x) && (5.11) \\ &= A'\cos(x) + B'\sin(x) && (5.12) \end{aligned}$$

Thus, the real solution is

$$y = x^2 \sin(x) + A'\cos(x) + B'\sin(x) \tag{5.13}$$

6 Problem Six

[Boas, Ch.8, Sec.7, Problem 3]

Solve the following differential equation by method (a) or (b) from Sec.(7) in Boas:

$$2y\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2 \tag{6.1}$$

6.1 Solution Six

To solve this differential equation, we use method (b), since the independent variable x is missing. That is, we set

$$v(x) = \frac{dy}{dx} \tag{6.2}$$

and

$$\frac{dv}{dx} = \frac{dv}{dy}\frac{dy}{dx} = v\frac{dv}{dy} \tag{6.3}$$

Plugging these into the differential equation, we find a new, seperable differential equation for $v(y)$:

$$2yv\frac{dv}{dy} = v^2 \tag{6.4}$$

Seperating the differential equation and integrating both sides, we find that

$$\ln(v) = \ln(\sqrt{y}) + C \qquad \Longrightarrow \qquad \frac{dy}{dx} = v = C\sqrt{y} \tag{6.5}$$

Now to find $y(x)$, we integrate the above result obtaining:

$$y = \frac{1}{4}(Cx + D)^2 \tag{6.6}$$

7 Problem Seven

[Boas, Ch.8, Sec.7, Problem 5]

The differential equation of a hanging chain supported at its ends is

$$\left(\frac{d^2y}{dx^2}\right)^2 = k^2\left(1+\left(\frac{dy}{dx}\right)^2\right) \tag{7.1}$$

Solve the equation to find the shape of the chain.

7.1 Solution Seven

To solve this equation, we let $dy/dx = v$. Then, we have

$$\frac{d^2y}{dx^2} = \frac{dv}{dx} \tag{7.2}$$

Plugging this in, we find

$$\left(\frac{dv}{dx}\right)^2 = k^2\left(1+v^2\right) \tag{7.3}$$

Taking the squared root, we find

$$\frac{dv}{dx} = \pm k\sqrt{1+v^2} = \eta k\sqrt{1+v^2} \tag{7.4}$$

Where $\eta = \pm 1$. Now let's integrate this equation

$$\int \frac{dv}{\sqrt{1+v^2}} = \eta k \int dx = \eta k x + D \tag{7.5}$$

To integrate the left-hand-side, we make the change of variable $v = \sinh(u)$. Note that $\cosh(u)^2 = 1 + \sinh^2(u)$

$$\int \frac{dv}{\sqrt{1+v^2}} = \int \frac{\cosh(u)du}{\sqrt{1+\sinh(u)^2}} = \int \frac{\cosh(u)du}{\cosh(u)} = u = \sinh^{-1}(v) \tag{7.6}$$

Therefore, we find

$$v = \sinh(\eta k x + D) = \frac{dy}{dx} \tag{7.7}$$

Integrating once more, we find

$$y(x) = \frac{1}{\eta k}\cosh(\eta k x + D) + C \tag{7.8}$$

Note that

$$\cosh(-kx + D) = \cosh(-(kx - D)) = \cosh(kx - D) = \cosh(kx + D') \tag{7.9}$$

Thus, the full solution is

$$y(x) = \pm\frac{1}{k}\cosh(kx + D) + C \tag{7.10}$$

8 Problem Eight

[Boas, Ch.8, Sec.7, Problem 11] 436.11

In following problem, solve (7.14) to find $v(x)$ and then $x(t)$ for the given $F(x)$ and initial conditions:

$$F(x) = -2m/x^5; \qquad v(t=0) = -1, x(t=0) = 1. \tag{8.1}$$

8.1 Solution Eight

Our differential equation is

$$m\frac{d^2y}{dx^2} = -2\frac{m}{x^5} \tag{8.2}$$

This can be written as $v' = -2x^{-5}$. Using

$$\frac{dv}{dt} = \frac{dx}{dt}\frac{dv}{dx} = v\frac{dv}{dx} \tag{8.3}$$

we find $v(dv/dx) = -2x^{-5}$. Separating, we have

$$vdv = -2\frac{dx}{x^5} \tag{8.4}$$

Integrating, we find

$$\frac{1}{2}v^2 = \frac{1}{2x^4} + \frac{1}{2}D \tag{8.5}$$

Solving for v, we have

$$v = \pm\sqrt{D + x^{-4}} = \pm\frac{\sqrt{Dx^4 + 1}}{x^2} \tag{8.6}$$

Given that $x(t=0) = 1, v(t=0) = -1$ we find at at $t = 0$:

$$-1 = -\frac{\sqrt{D+1}}{1} \qquad \Longrightarrow \qquad D = 0 \tag{8.7}$$

Thus, we have that

$$v(x) = \frac{dx}{dt} = -x^{-2} \tag{8.8}$$

Separating variables, we find

$$x^2dx = -dt \tag{8.9}$$

Integrating, we obtiain

$$\frac{1}{3}x^3 = -t + \frac{1}{3}C \tag{8.10}$$

Solving for x, we arrive at

$$x = (C - 3t)^{1/3} \tag{8.11}$$

Using $x(t = 0) = 1$, we find that $C = 1$. Thus, the final solution is

$$x(t) = (1 - 3t)^{1/3} \tag{8.12}$$

9 Problem Nine

[Boas, Ch.8, Sec.7, Problem 13] 436.13

The exact equation of motion of a simple pendulum is $d^2\theta/dt^2 = -\omega_0^2 \sin(\theta)$ where $\omega_0^2 = g/l$. By method (c) in Chap. 8, Sec.7 of Boas, integrate this equation once to find $d\theta/dt$ if $d\theta/dt = 0$ when $\theta = 90°$. Write a formula for $t(\theta)$ as an integral. See Problem 5.34.

9.1 Solution Nine

Let $\omega = d\theta/dt$. Then we have that

$$\frac{d\omega}{dt} = -\omega_0^2 \sin(\theta) \tag{9.1}$$

Using $d\omega/dt = \omega(d\omega/d\theta)$, this becomes

$$\omega\frac{d\omega}{d\theta} = -\omega_0^2 \sin(\theta) \tag{9.2}$$

Integrating both sides, we find

$$\frac{1}{2}\omega^2 = \omega_0^2 \cos(\theta) + \frac{1}{2}D \tag{9.3}$$

Solving for ω, we find

$$\omega = \frac{d\theta}{dt} = \sqrt{2\omega_0^2 \cos(\theta) + D} \tag{9.4}$$

Using the initial condition, we find that $D = 0$. Now we have that

$$\frac{d\theta}{dt} = \sqrt{2}\omega_0\sqrt{\cos(\theta)} \tag{9.5}$$

Separating variables,

$$\frac{d\theta}{\sqrt{\cos(\theta)}} = \sqrt{2}\omega_0 dt \tag{9.6}$$

Integrating both sides, we find that

$$t = \frac{1}{\sqrt{2}\omega_0}\int \frac{d\theta}{\sqrt{\cos(\theta)}} \tag{9.7}$$

10 Problem Ten

[Boas, Ch.8, Sec.7, Problem 26]

For the following problem, verify the given solution and then, by method (e) from Chap.8, Sec.7 of Boas, find a second solution of the given equation.

$$(x^2+1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0, \qquad u = x. \tag{10.1}$$

10.1 Solution Ten

We are told that $u = x$ is a solution. We can easily check this:

$$(x^2+1)\frac{d^2u}{dx^2} - 2x\frac{du}{dx} + 2u = (x^2+1)\times 0 - 2x \times 1 + 2x = -2x + 2x = 0 \tag{10.2}$$

To solve for the second solution, we set $y = u(x)v(x)$. Doing so, we find the following differential equation for $v(x)$:

$$x(1+x^2)\frac{d^2v}{dx^2} + 2\frac{dv}{dx} = 0 \tag{10.3}$$

This is just a first order differential equation, which we can see by setting $h(x) = dv/dx$:

$$x(1+x^2)\frac{dh}{dx} + 2h(x) = 0 \tag{10.4}$$

This is a seperable differential equation. Separating variables, we obtain

$$\frac{dh}{h} = -\frac{2dx}{x(1+x^2)} \tag{10.5}$$

Integrating, we find

$$\ln(h) = \log\left(\frac{1+x^2}{x^2}\right) + C \tag{10.6}$$

Exponentiating, we find that

$$\frac{dv}{dx} = h(x) = C'\left(\frac{1+x^2}{x^2}\right) \tag{10.7}$$

Integrating once more to find $v(x)$, we arrive at

$$v(x) = C'\left(x - \frac{1}{x}\right) + D \tag{10.8}$$

Thus, our second solution is

$$y(x) = u(x)v(x) = C'\left(x^2 - 1\right) + Dx \tag{10.9}$$

which is in fact our entire solution.