Physics 116B- Spring 2018

Mathematical Methods 116 B

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Notes on Complex-Functions-I

§Complex Functions: What are they :

Transiting from real to complex is mainly straightforward:

$$x \to z = (x + iy)$$

Polar form

$$z = \rho e^{i\phi}, \;\; x = \rho \cos(\phi), \; y = \rho \sin(\phi).$$

When ϕ winds through 2π z returns to its original value.

$$e^{i2\pi} = 1.$$

$$f(x) \to f(z)$$

We have seen $e^z, \sin(z), \cos(z), \tanh(z)$ etc. These are complex functions Complex functions have real and imaginary parts.

$$f(z) = u(x, y) + iv(x, y)$$

and we say that $u = \Re e f(z)$ and $v = \Im m f(z)$.

§Complex Functions: Properties of interest: Single versus multivalued functions:

Single valued example:

$$z^2 = (x + iy)^2 = (x^2 - v^2) + i2xy,$$

$$e^z$$

Multivalued:

$$z^{\frac{1}{2}} = \sqrt{\rho}e^{\frac{1}{2}i\phi}$$

where $\phi \to \phi + 2\pi$ does not return z to its old value, we need to wind around *twice* to get back.

This is said to have a square root *branch point*.

$$\log z = \log \rho + i\phi,$$

which never returns on winding around. Infinite fold branch point.

§**Complex Functions: Analytic functions:**

Can we differentiate a complex function uniquely?

This is a **key question**.

Quick reminder for real functions where the derivative at a point x is defined by

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

Since x is real Δx must also be real. It can be positive or negative. If we get the same answer from both sides, we say it has a unique derivative at that point.

Similarly for complex functions we may define:

$$\frac{df}{dz} = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}. \tag{1}$$

For complex functions the *key point* is that a corresponding Δz can be one of many things.

e.g.

$$\Delta z = \Delta x$$

$$\Delta z = i\Delta y$$

$$\Delta z = p\ \Delta x + iq\ \Delta y$$

where p, q are themselves complex numbers.

Pictorially this means we can wander away from any z in an infinite number of directions.

If the derivative is unique, that would be special....it is special and leads to the theory of analytic functions.

(i) Positive example.

$$f(z) = z^2 = (x+iy)^2$$

Let us try

$$\Delta z = \Delta x$$

$$f(z+\Delta x) = (x+\Delta x+iy)^2 \sim (x+iy)^2 + 2(x+iy)\Delta x$$

Hence

$$\lim_{\Delta x \to 0} \frac{f(z+\Delta x) - f(z)}{\Delta x} \to 2z$$

Similarly with $\Delta z = i\Delta y$

$$f(z+i\Delta y) = (x+iy+i\Delta y)^2 \sim (x+iy)^2 + 2i\Delta y(x+iy)$$

Hence we get the same answer as above.

$$\lim_{\Delta y \to 0} \frac{f(z+i\Delta y) - f(z)}{i\Delta y} \to 2z$$

(ii) Negative example

$$f(z) = |z| = \sqrt{x^2+y^2}$$

Let us try

$$\Delta z = \Delta x$$

$$f(z+\Delta x) = \sqrt{(x+\Delta x)^2 + y^2} \sim \sqrt{x^2 + 2x\Delta x + y^2} \sim \left(|z| + \frac{x\Delta x}{|z|} + o(\Delta x)\right)$$

Hence

$$\lim_{\Delta x \to 0} \frac{f(z+\Delta x) - f(z)}{\Delta x} \to \frac{x}{|z|}$$

Similarly with $\Delta z = i\Delta y$

$$f(z+i\Delta y) = \sqrt{(y+i\Delta y)^2 + x^2} \sim \sqrt{y^2 + 2iy\Delta y + x^2} \sim \left(|z| + \frac{iy\Delta y}{|z|} + o(\Delta y)\right)$$

Here we get a different answer from above

$$\lim_{\Delta y \to 0} \frac{f(z+i\Delta y) - f(z)}{i\Delta y} \to \frac{y}{|z|}$$

Hence in the positive case we verified that

$$\frac{df(z)}{dz}$$

can be calculated by any (small) variation of z, and we get the same answer. This defines an *analytic function.*

In the negative case we get a different answer, so it is *a non-analytic function.*

Recap

§Definition of an analytic function:

Given a function $f(z_0)$, we define its derivative at the point z_0 as

$$f'(z_0) = \left.\frac{df(z)}{dz}\right|_{z_0} = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}. \tag{2}$$

In general dz can take an infinite variety of "directions", this is a property of 2-dimensions, and corresponds to approaching the point z_0 in one of ∞ ways.

:
See:Fig:
If

$$\left.\frac{df(z)}{dz}\right|_{z \longrightarrow z_0}$$

obtained by a calculation with an arbitrary variation dz gives the same answer it is an *analytic function* at the point z_0.

Corollary In case we get a different answer depending on the variation dz, it is *a non-analytic function.*

Comment We can have a function that is analytic at z_0 and not at z_1
Example:

$$f(z) = \frac{1}{\sin(z)}, \quad \text{or} \quad \frac{1}{z}$$

are analytic at $z = 1$ but not at $z = 0$.

We will see soon that these functions are said to have a pole at $z = 0$, it is a kind of a singularity of the function f.

§Cauchy-Riemann theorem:

One of the very important theorems:

If $f(z)$ is analytic at some point z, and if $f(z) = u(x, y) + iv(x, y)$ with real u, v, then u, v satisfy the Cauchy-Riemann conditions

$$\partial_x u = \partial_y v, \quad \partial_x v = -\partial_y u, \tag{3}$$

Note we will also write the partial derivatives more compactly for any function $a = a(x, y)$ as

$$\partial_x a = a_x, \quad \partial_y a = a_y.$$

Proof:
Consider $dz = dx$

$$f'(z) = \frac{d(u + iv)}{dx}\Big|_{y \text{ fixed}} = u_x + iv_x$$

Now $dz = idy$

$$f'(z) = \frac{d(u + iv)}{idy}\Big|_{x \text{ fixed}} = -iu_y + v_y$$

Equating

$$u_x + iv_x = -iu_y + v_y,$$

Hence separately equating real and imaginary parts we get the required result

$$u_x = v_y, \tag{4}$$

$$u_y = -v_x. \tag{5}$$

The reverse is also true, if Cauchy-Riemann conditions are true, the function is analytic. We skip the proof.

§**A simple consequence: u, v satisfy Laplace's equation in 2-d.:**

Take x derivative of Eq. (4) and y derivative of Eq. (5) and add. since $v_{xy} = v_{yx}$

$$\nabla^2 u = u_{xx} + u_{yy} = 0 \tag{6}$$

$$\nabla^2 v = v_{xx} + v_{yy} = 0. \tag{7}$$

These are Laplace's equations that arise in electrostatics, where the potential satisfies this equation in a region where there are no source of charge.

Hence solution of most electrostatic problems in 2-d can be done using complex functions.

Let us see a few examples.

$$f(z) = z^2$$

$$f(z) = e^z$$

Given one real solution u of Laplace's equation, we can find its complementary solution v by finding an analytic function $f = u + iv$. Mostly this is easy by guessing:

$$u = 3x^2y - y^3$$

$$v_y = u_x = 6xy \quad v_x = -u_y = 3y^2 - 3x^2$$

Integrate the first of these

$$v = 3xy^2 + a(x)$$

Now plug into second

$$a'(x) + 3y^2 = 3y^2 - 3x^2$$

Hence

$$a = -x^3$$

Hence

$$v = 3xy^2 - x^3.$$

Thus we have a complex function

$$f(z) = (3x^2y - y^3) + i(3xy^2 - x^3)$$

or

$$f(z) = -iz^3.$$

§Isolated singularities and regular points:

$$f(z) = \frac{1}{z-a}, \quad f(z) = \frac{1}{(z-a)^2},$$

are examples of functions with isolated singular points- in tis case poles of degree 1 and 2 respectively.

Here $f(z)$ is analytic in a domain D which excludes the points $z = a$. Pictured as below.

We can also have an infinite number of poles.

$$f(z) = \cot(z) = \frac{\cos(z)}{\sin z} = \frac{1}{z} + \sum_{n \neq 0} \left(\frac{1}{z - n\pi} + \frac{1}{n\pi} \right)$$

Here $f(z)$ is analytic in a domain D which excludes the points $z = 0, \pm n\pi$. Pictured as below.

We will learn more about various types of singularities soon.

§**Another important pair of theorems:**

(I) If $f(z)$ is analytic in a domain D, then **all its derivatives** *are also analytic, i.e.* $\frac{d^n(z)}{dz^n}$ *are analytic.*

(II) If $f(z)$ is analytic in a domain D, then it can be Taylor expanded about any point z_0 in the domain. The Taylor series converges inside a circle around z_0 of a diameter that is given by the location of the nearest singularity.

This is a very useful theorem even for real series.

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots$$

Convergence circle is $|x| = 1$. Why is it so? We can now answer it.

§ **Big theorem: Cauchy's theorem of integral of an analytic function:** If f is analytic in a domain D, the closed contour integral of f vanishes, provided the contour Γ lies entirely in D.

$$\oint_\Gamma f(z)\, dz = 0$$

§**A very useful warmup exercise:**

We might wonder why the big fuss about zero! To lead up to this let us do a few integrals.

$$f(z) = z^n$$

and the contour Γ is a circle of radius R.

$$z = Re^{i\phi}, \;\; dz = iRe^{i\phi}\, d\phi$$

with

$$0 \le \phi < 2\pi$$

Hence :

$$\oint_\Gamma f(z)\, dz = \int_0^{2\pi} iRe^{i\phi}\, d\phi \times R^n e^{in\phi} = iR^{n+1} \int_0^{2\pi} e^{i(n+1)\phi} d\phi$$

The answer is

$$\oint_\Gamma f(z)\, dz = \frac{R^{n+1}}{n+1} \left(e^{i(n+1)2\pi} - 1\right) = 0.$$

Vanishes regardless of R and also n. That is truly remarkable!
One other case, let n become negative. If $n \neq -1$ the same answer follows.
However if $n = -1$ we have a special case that is worth understanding.

$$\oint_\Gamma \frac{1}{z}\, dz = \int_0^{2\pi} iRe^{i\phi}\, d\phi \times \frac{1}{Re^{i\phi}} = i \int_0^{2\pi} d\phi = 2\pi i.$$

This is also remarkable, it is non zero and is independent of R. This means some non-analytic functions give interesting results on integration. Let us get to the bottom of this.

We summarize the results For Γ a circle of radius R around the origin, the integral for all n, both positive and negative, is calculated to be :

$$\oint_\Gamma z^n\, dz = (2\pi i)\delta_{n,-1}$$

Cauchy theorem for Taylor expandable functions is clearly working here. If I take a function $f(z) = \sum_n c_n z^n$ then its integral is also zero!!

§**Proof of Cauchy's theorem:** We compute the contour integral of an

analytic function $f(z)$, where the contour Γ lies within D the domain of its analyticity. We write

$$\mathcal{I} = \oint_\Gamma f(z)\, dz = \oint_\Gamma \left(u(x,y) + iv(x,y)\right)\, (dx + idy).$$

By separating real and imaginary parts

$$\mathcal{I} = \oint_\Gamma (udx - vdy) + i \oint_\Gamma (vdx + udy)\,.$$

Use Greens theorem

$$\oint_\Gamma (Adx + Bdy) = \int_S dx\, dy\, (B_x - A_y)$$

where S is the surface area bounded by Γ and the directions are as in figure:

Figure:

Let us use Greens theorem and the Cauchy Riemann conditions, (dropping the symbols Γ, S)

$$\int (udx - vdy) \rightarrow \int dxdy\,(v_x + u_y) \rightarrow 0$$

$$\int (vdx + udy) \rightarrow \int dxdy\,(u_x - v_y) \rightarrow 0$$