Physics 116B- Spring 2018

Mathematical Methods 116 B

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Notes on Complex-Functions-II

§Scheme for working with Complex functions:

- Let us consider a domain D, e.g. some closed region in complex plane. Inside it:
- Given that $f(z)$ is analytic, its real and imaginary parts $f = u + iv$ have certain properties.
- $u_x = v_y$ and $u_y = -v_x$ Cauchy Riemann conditions,
- Both satisfy the Laplace equation $\nabla^2 u = 0$ and $\nabla v = 0$, where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

These also work in reverse as follows:

- Given a real function $u(x, y)$ The question: Is u part of an analytic complex function $f(z)$?
- This question is important since if the answer is yes, we can get a lot of results with little work.
- First check if it satisfies Laplace's equation. If it does not the answer is no. If yes we can dig deeper and get f
- Find v using Cauchy-Riemann equations
- Reconstruct $f = u + iv$.

§ **Warm up problems with complex functions:**

Problem: Take two closely related functions

$$u_a = x^2 + y^2, \;\; u_b = x^2 - y^2,$$

and check if they are the real parts of an analytic function. If yes, find that $f(z)$.

Problem: #61/674
Given

$$u(x,y) = \frac{x}{x^2 + y^2},$$

show that it is a harmonic function satisfying the Laplace equation, and hence can be viewed as the real part of a analytic function $f(z)$. Find the partner function $v(x,y)$ from the Cauchy Riemann conditions.

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We saw in the last class that

$$\oint_\Gamma z^n \, dz = 0, \quad \text{unless } n = -1.$$

where Γ is a circle of radius R around the origin.

New Problem:

Calculate

$$\oint_\Gamma \bar{z}^n \, dz$$

where Γ is a circle of radius R around the origin and $\bar{z} = z^*$, for $n = 1, 2$.

Problem: #3/676

Calculate

$$\oint_\Gamma z^2\, dz$$

where

1) Γ is a closed contour in the form of a semi-circle closed by the real line.

2) Γ is a rectangular path with corners at $z = -1, 1, 1+i, -1+i$

§**Proof of Cauchy's theorem:** We compute the contour integral of an analytic function $f(z)$, where the contour Γ lies within D the domain of its analyticity.

Note: Contour is the same thing a closed curve. What I callas a closed contour Γ os what Boas calls it a closed curve C.

We write

$$\mathcal{I} = \oint_\Gamma f(z)\, dz = \oint_\Gamma (u(x,y) + iv(x,y))\ (dx + idy).$$

By separating real and imaginary parts

$$\mathcal{I} = \oint_\Gamma (udx - vdy) + i \oint_\Gamma (vdx + udy)\,.$$

Use Greens theorem. It equates a line integral to a surface integral for a simply connected domain

$$\oint_\Gamma (Adx + Bdy) = \int_S dx\, dy\, (B_x - A_y)$$

where S is the surface area bounded by Γ and the direction of going around is counterclockwise, as in figure:

Figure:

Let us use Greens theorem and the Cauchy Riemann conditions, (dropping the symbols Γ, S)

$$\int (udx - vdy) \to \int dxdy\,(v_x + u_y) \to 0$$

$$\int (vdx + udy) \to \int dxdy\,(u_x - v_y) \to 0$$

§ **Cauchy's integral formula:**

If $f(z)$ is analytic inside a domain D, i.e. for the same conditions as for the earlier theorem and a is some point inside the domain.

$$\oint_\Gamma \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$$

Let us first see what it means or implies, and then prove it.

Remarks: Value of a (provided it is in the interior of D) and Γ are arbitrary.

Example of two circles of radii $R_1 > R_2$.

Pictorial proof.

§**Some problems to get used to complex integrals.:**

\# 1/676
Calculate

$$\int_{i}^{1+i} z\,dz$$

along a path parallel to the real axis.

\# 11/677
Calculate

$$\oint_{\Gamma} (\bar{z} - 3)\,dz$$

where Γ is closed path traversing $(0, 2)$ along real line, $(2, 2i)$ on a semicircle, $(2i, 0)$ on the y axis.

Problem # 14/677

Show

$$\int_0^{2\pi} e^{inx} e^{-imx}\, dx = 0, \quad n \neq m$$

from applying Cauchy's theorem to

$$\oint_\Gamma z^{n-m-1}\, dz, \quad n > m$$

where Γ is the unit circle.

Problem # 18/677 Calculate

$$\oint_\Gamma \frac{\sin(2z)}{6z-\pi}\,dz,$$

where Γ is a circle $|z| = 3$.

§ **Connecting Cauchy's integral formula to the Taylor expansion:**

We will need the simple formula below. Let us record it here.

$$\frac{d}{dw}\frac{1}{(z-w)^m} = \frac{m}{(z-w)^{m+1}}. \quad \text{(Formula-1)}$$

We saw the Cauchy integral formula: If $f(z)$ is analytic inside a domain D, i.e. for the same conditions as for the earlier theorem and a is some point inside the domain, and Γ is a closed contour surrounding the point a lying inside the domain D, then

$$\oint_\Gamma \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$$

Let us switch variables and rewrite this more suggestively as

$$f(w) = \oint_\Gamma \frac{f(z)}{z-w}\frac{dz}{2\pi i},$$

where w is another name for a.

Now take a derivative of both sides w.r.t. w using the Formula-1 for $m = 1$.

$$f'(w) = \oint_\Gamma \frac{f(z)}{(z-w)^2}\frac{dz}{2\pi i},$$

Take one more derivative:

$$f^{(2)}(w) = 2\oint_\Gamma \frac{f(z)}{(z-w)^3}\frac{dz}{2\pi i},$$

etc. Hence

$$f^{(n)}(w) = n!\oint_\Gamma \frac{f(z)}{(z-w)^{n+1}}\frac{dz}{2\pi i},$$

We can "deduce" the Taylor expansion from this formula

$$f(z) = \sum_{n=0}^{\infty} (z-w)^n \frac{1}{n!} f^{(n)}(w)$$

The Taylor series converges as long as both w, z are in the domain D. For this we need a result we have already proven

$$\oint_{\Gamma} (z-w)^m \frac{dz}{2\pi i} = \delta_{m,-1}.$$