Physics 116B- Spring 2018

Mathematical Methods 116 B

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Notes on Complex-Functions-III

Problem: #3/676
Calculate

$$\oint_{\Gamma} z^2 \, dz$$

where
1) Γ is a closed contour in the form of a semi-circle closed by the real line.
2) Γ is a rectangular path with corners at $z = -1, 1, 1+i, -1+i$

Problem # 18/677 Calculate

$$\oint_{\Gamma} \frac{\sin(2z)}{6z-\pi}\, dz,$$

where Γ is a circle $|z| = 3$.

§**Single valued and multivalued functions and poles versus branch points:**

In a domain D a function can be either of these.

- Single valued:Examples

  Non-singular: Called entire functions:

  $$e^z, \quad or \quad \sin(z)$$

  Singular: With simple pole

  $$\frac{1}{z-w}$$

  : Simple pole at $z = w$ of degree one. Also called meromorphic.

  Singular: With simple poles

  $$\frac{1}{z-w_1}\frac{1}{z-w_2}\frac{1}{z-w_3}$$

  : Simple poles at $z = w_j$ of degree one. Also called meromorphic.

  Singular: With poles of higher degree:

  $$\frac{1}{(z-w)^n}$$

  Pole of degree n.

- Multivalued: Example

  $$\log z \quad or \quad \sqrt{z}$$

  . We can do some book-keeping and make them single valued.

$\sqrt{\mathbf{z}}$

$\log \mathbf{z}$

§ **Cauchy's residue theorem :**

If $f(z)$ is meromorphic (i.e. analytic except for a set of poles at $w_1, w_2, \ldots w_m$) in the domain D and with a counter-clockwise Γ lying inside this domain

$$\mathcal{I} = \oint_\Gamma f(z)\, dz = 2\pi i \sum_{j=1,m} \text{Residue}(j)$$

Here $Residue_j$ at pole $z = w_j$ of degree n is defined as

$$R = \frac{1}{(n-1)!} \lim_{z \to w_j} \frac{d^{n-1}}{dz^{n-1}} (z - w_j)^n f(z).$$

Simple examples:

Γ is a circle around origin of radius 3

$$f = \frac{1}{z - 4 - i}$$

Γ is a circle around origin of radius 3

$$f = \frac{\sin(z)}{(z-4)(z+i)}$$

§ **Connecting Cauchy's integral formula to the Taylor expansion:**

We will need the simple formula below. Let us record it here.

$$\frac{d}{dw}\frac{1}{(z-w)^m} = \frac{m}{(z-w)^{m+1}}. \quad \text{(Formula-1)}$$

We saw the Cauchy integral formula: If $f(z)$ is analytic inside a domain D, i.e. for the same conditions as for the earlier theorem and a is some point inside the domain, and Γ is a closed contour surrounding the point a lying inside the domain D, then

$$\oint_\Gamma \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$$

Let us switch variables and rewrite this more suggestively as

$$f(w) = \oint_\Gamma \frac{f(z)}{z-w}\frac{dz}{2\pi i},$$

where w is another name for a.

Now take a derivative of both sides w.r.t. w using the Formula-1 for $m = 1$.

$$f'(w) = \oint_\Gamma \frac{f(z)}{(z-w)^2}\frac{dz}{2\pi i},$$

Take one more derivative:

$$f^{(2)}(w) = 2\oint_\Gamma \frac{f(z)}{(z-w)^3}\frac{dz}{2\pi i},$$

etc. Hence

$$f^{(n)}(w) = n!\oint_\Gamma \frac{f(z)}{(z-w)^{n+1}}\frac{dz}{2\pi i},$$

We can "deduce" the Taylor expansion from this formula

$$f(z) = \sum_{n=0}^{\infty} (z-w)^n \frac{1}{n!} f^{(n)}(w)$$

The Taylor series converges as long as both w, z are in the domain D. For this we need a result we have already proven

$$\oint_{\Gamma} (z-w)^m \frac{dz}{2\pi i} = \delta_{m,-1}.$$

§**Laurent expansion or Laurent series by example:**

Let us call C_x as a circle of radius x.

First example: Here f is analytic in a region between C_r with $r \ll 1$, and C_1.

$$\begin{aligned} f(z) &= \frac{1}{z(1-z)} = \frac{1}{z}(1 + z + z^2 + z^3 + z^4 + z^5 + \ldots), \\ &= \frac{1}{z} + 1 + z + z^2 + \ldots \qquad (1) \end{aligned}$$

If we leave the first term out, the second term onwards converges for $|z| < 1$ from the ratio test.

This is an example of the Laurent series. Note it has positive and negative exponents, and is convergent in the stated region.

The Laurent series depends on the region where we are expanding the function.

To see this take same function and expand for $|z| > 1$, i.e. between C_1 and C_∞ We can write it more conveniently in this region as

$$
\begin{aligned}
f(z) &= -\frac{1}{z^2(1-1/z)} = \frac{1}{z^2}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\ldots\right) \\
&= -\left(\frac{1}{z^3}+\frac{1}{z^4}+\frac{1}{z^5}+\ldots\right) \qquad (2)
\end{aligned}
$$

Compare with other series for the same function valid for $|z| < 1$,

$$
f(z) = \frac{1}{z} + 1 + z + z^2 + \ldots
$$

Second example:

Find a convergent (Laurent) expansion in the region $0 < |z| < R$ and determine R.

$$
f(z) = \frac{z+1}{z^3(z^2+1)}
$$

This function has poles at $z = 0, \pm i$, and is analytic otherwise. We can expand around $z = 0$ as follows:

$$
\begin{aligned}
f(z) &= \left(\frac{1}{z^2}+\frac{1}{z^3}\right)\frac{1}{(z^2+1)} \\
&= \left(\frac{1}{z^2}+\frac{1}{z^3}\right)(1-z^2+z^4-z^6+\ldots) \\
&= \frac{1}{z^3}+\frac{1}{z^2}-\frac{1}{z}+z+z^2-z^3-z^4+\ldots \qquad (3)
\end{aligned}
$$

§**Laurent expansion or Laurent series formal statement:** In a region between two circles C_r and $C_{r'}$ around point z_0 where f is analytic, it can be expanded in a convergent Laurent expansion

$$\begin{aligned} f(z) &= a_0 + \sum_{n=1,2,\ldots} a_n (z-z_0)^n \\ &+ \sum_{n=1,2,\ldots} b_n (z-z_0)^{-n}. \end{aligned} \tag{4}$$

The coefficients can be calculated from the residue theorem as

$$\begin{aligned} a_n &= \frac{1}{2\pi i} \oint_C \frac{f(z)dz}{(z-z_0)^{n+1}} \\ b_n &= \frac{1}{2\pi i} \oint_C \frac{f(z)dz}{(z-z_0)^{-n+1}}. \end{aligned} \tag{5}$$

Easy to show by plugging in the Laurent expansion in Eq. (1) into Eq. (2).

With $A > 1$ calculate

$$I = \int_0^{2\pi} \frac{d\theta}{A + \cos(\theta)}.$$

Convert to contour integral as follows:

$$z = e^{i\theta}, \;\; dz = ie^{i\theta} d\theta, \;\; \text{i.e.} \;\; dz = iz d\theta.$$

Integral corresponds to a full circle of radius 1, i.e. C_1. Hence

$$I = \frac{1}{i} \oint_{C_1} \frac{z^{-1}}{A + \frac{1}{2}(z + z^{-1})} dz = \frac{2}{i} \oint_{C_1} \frac{1}{1 + 2Az + z^2} dz$$

Simplifying and calling $a_\pm = -A \pm \sqrt{A^2 - 1}$

$$I = \frac{2}{i} \oint_{C_1} \frac{1}{(z - a_+)(z - a_-)}.$$

Now since $|a_-| > |a_+|$ inside the circle we have one pole, and one pole is outside the circle. Residue is simple: Result

$$I = 2\pi \frac{1}{\sqrt{A^2 - 1}}.$$

§ **Cauchy's residue theorem Revisited: Higher order poles:**

If $f(z)$ is meromorphic (i.e. analytic except for a set of poles at $w_1, w_2, \ldots w_m$) in the domain D and with a counter-clockwise Γ lying inside this domain

$$\begin{aligned} \mathcal{I} &= \oint_\Gamma f(z)\, dz = \sum_{j=1,m} \mathcal{I}_j \\ \mathcal{I}_j &= 2\pi i\, R_j \end{aligned} \tag{6}$$

Here R_j is the *residue* at pole $z = w_j$ of degree n. It is defined as

$$R_j = \frac{1}{(n-1)!} \lim_{z \to w_j} \frac{d^{n-1}}{dz^{n-1}} (z - w_j)^n f(z).$$

A good way to think about this theorem,

- Find or locate the m poles. Make sure they are isolated poles, i.e. exclude things like $\log(z)$ which has no isolated singularity.
- Deform the contour Γ into a sum over circles of radius $R \to 0$ around each pole w_j. Yes, this can be done! Use the Cauchy theorem to add the smaller closed countours needed for this- see example.
- At each pole relabel things to make life easier. Call $z = w_j + v$ and write $dz = dv$, The contour in the v variable is now a circle C_R. The function can be rewritten after translation as as

  $$f(z) = f(w_j + v) \equiv g(v).$$

  The contribution $\mathcal{I}_j$ from this pole is now

  $$\mathcal{I}_j = \oint_{C_R} dv\; g(v).$$

- At each pole we now have a simpler problem, where we can use the Laurent (Taylor) expansion:

  $$g(v) = b_n \frac{1}{v^n} + b_{n-1} \frac{1}{v^{n-1}} + \ldots.$$

- Use the *basic result* $\oint_{C_R} \frac{dv}{v^m} = 2\pi i \delta_{m,-1}$.
- Hence $\mathcal{I}_j = 2\pi i b_1$

Problems

$$\mathcal{I} = \oint_{C_1} dz\, \frac{e^{3z}}{z^3}$$

By residue theorem $\mathcal{I} = 2\pi i R_0$.
Residue at $z = 0$ of $\frac{e^{3z}}{z^3}$.
Use strategy described. We should expand

$$\frac{e^{3z}}{z^3} = \frac{1}{z^3} + \frac{3}{z^2} + \frac{9}{2z} + \dots$$

We can zoom into the needed term and throw out the rest. Hence answer is

$$\mathcal{I} = 2\pi i \times \frac{9}{2}.$$

Check from formula

$$R_j = \frac{1}{(n-1)!} \lim_{z \to w_j} \frac{d^{n-1}}{dz^{n-1}} (z - w_j)^n f(z)$$

Here $n = 3, w_j = 0$, and

$$f(z) = \frac{e^{3z}}{z^3}.$$

$$R_j = \frac{1}{2!} \lim_{z \to 0} \frac{d^2}{dz^2} (z^3 f(z)) = \frac{9}{2}.$$

§

$$\mathcal{I} = \oint_{C_1} dz \ \cot^2(z)$$

§

$$\mathcal{I} = \oint_{C_1} dz \ z \cot^2(z)$$

Problem # 14/677
Show

$$\int_0^{2\pi} e^{inx} e^{-imx} \, dx = 0, \ \ n \neq m$$

from applying Cauchy's theorem to

$$\oint_{\Gamma} z^{n-m-1} \, dz, \ \ n > m$$

where Γ is the unit circle. For $n = m$ show that the result is 2π.

Residue problems

Residue at $z = 1$ of

$$\frac{e^z}{z^2 - 1}$$

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Residue at $z = \pi$ of

$$\frac{1 + \cos(z)}{(z - \pi)^2}$$

#8/686

Calculate

$$I = \int_{-\infty}^{\infty} \frac{dx}{a^2 + x^2}$$