Physics 116B- Spring 2018

Mathematical Methods 116 A

S. Shastry, April 17, 2018
Notes on Differential Equations-II

§Solution of a simple non linear ODE of first order

$$\frac{dy}{dt} = -ay^3 \tag{1}$$

We expect that a single IC should suffice since this is a first order ODE.

This is like the population decay problem

$$\frac{dy}{dt} = -ay,$$

with solution

$$y(t) = y_0 e^{-at}, \tag{2}$$

but with a different power of y on the right hand side.

Solution: Since the RHS involves only y and not t, we can use the separation idea

$$\begin{aligned}
\frac{dy}{y^3} &= -a\, dt \\
\int_{y_0}^{y} \frac{dy}{y^3} &= -a \int_0^t dt \\
\frac{1}{2}\left(\frac{1}{y_0^2} - \frac{1}{y^2}\right) &= -a\, t
\end{aligned} \tag{3}$$

The solution can be rewritten as

$$y(t) = y_0 \left(\frac{1}{1 + 2a\, t\, y_0^2}\right)^{\frac{1}{2}} \tag{4}$$

§**Physical interpretation of solution:**

At very long times we see that the solution Eq. (4) behaves as

$$y(t) \sim \frac{1}{\sqrt{2at}},$$

which is independent of y_0 and much slower than Eq. (2), the population problem solution.

Another remarkable feature of this non-linear equation is found if we switch the sign of $a \to -|a|$. Observe that Eq. (4) will now **diverge** at a finite time

$$t^* = \frac{1}{2|a|y_0^2}. \tag{5}$$

The population problem Eq. (2) with this sign of a will also diverge, but at $t \to \infty$ instead of a finite time t^*.

§**Another non-linear but separable ODE:**

Another example of a separable non linear equation is as follows: §MB 398. 3

$$y' \sin x = y \log y,$$

with one condition $y = e$ when $x = \pi/3$.

Using separation we write

$$\frac{dy}{y \log y} = \frac{dx}{\sin x},$$

which can be further simplified by using $\phi = \log y$ so that $d\phi = dy/y$ and hence

$$\frac{d\phi}{\phi} = d(\log \phi) = \frac{dx}{\sin x},$$

Now we calculate, on a separate line the integral on right. We verify

$$d \log \tan x/2 = dx/ \sin x,$$

and hence

$$d(\log \phi) = d(\log \tan x/2),$$

and hence

$$\log \phi = \log \tan x/2 + A$$

or

$$\phi = A' \tan x/2$$

with $A' = e^A$, thus

$$y = e^{A' \tan x/2}.$$

We now use the initial condition at $x = \pi/3$. Note that $\tan \pi/6 = 1/\sqrt{3}$ and hence we get $y = e$ provided

$$A' = \sqrt{3}.$$

Hence the particular solution required is

$$y = e^{\sqrt{3} \tan x/2}.$$

§ **Harmonic oscillator:**

Let us study the general solution of a 2nd order ODE

$$\ddot{y} = -\omega^2 y,$$

which can be integrated immediately as

$$\begin{aligned} y &= Ae^{i\omega t} + Be^{-i\omega t} \\ &= A' \cos \omega t + B' \sin \omega t \\ &= C \sin(\omega t + \phi). \end{aligned} \tag{6}$$

All three forms are equivalent. In the last form, ϕ is the "phase shift".

In any of these forms of the *general solution*, we see that there are two constants, say C, ϕ which need to be fixed. We can fix them to get a *particular solution*. For example we could say $y = 0$ at $t = 0$, thereby fixing $\phi = 0$. ($\phi = \pi$ leads to the same answer since $\sin(x + \pi) = -\sin(x)$ and we can absorb the sign into C.) We could then say that $\dot{y}|_{t=0} = 1$ which would fix $C\omega = 1$.

§**Particle dropped from a height h under gravity:**

This is an example of Newton's laws. A particle of mass m is dropped from a height h measured from the earth, and falls to the ground at a later time. We want to calculate the time it takes to fall down.

Set up equation: A few initial comments. We will imagine the x axis along the ground and the y axis increasing upwards. The dropping mass has a decreasing height, so the velocity $v < 0$ and also the acceleration $\dot{v} < 0$.

Newton's law gives

$$m\ddot{y} = -mg,$$

with $y(0) = h$. To repeat the convention of signs: the height y decreases as time progresses, hence the velocity and acceleration are both negative. Canceling m and integrating once

$$\dot{y}(t) = \dot{y}(0) - gt$$

Since we dropped the particle (rather than flinging it down or up), the initial velocity is zero, hence we can drop the term $\dot{y}(0)$ and get

$$\dot{y}(t) = -gt$$

so that integrating one more time

$$y(t) = -\frac{gt^2}{2} + y(0),$$

with $y(0) = h$, the initial height.

This is the solution, and is valid until the particle hits the ground. That happens when $y = 0$, i.e. $t = \sqrt{2h/g}$.

§ **Linear first order equations and integrating factors.:**

We now learn the systematic way of solving two related problems

$$\begin{aligned} y' + P(x)y &= 0 \qquad (7) \\ y' + P(x)y &= Q(x) \qquad (8) \end{aligned}$$

these are respectively, homogeneous and inhomogeneous equations.

§**Homogeneous equation**

We use the separation of variables and write

$$\frac{dy}{y} = -P(x)\,dx \qquad (9)$$

Let us call the indefinite integral $R(x) = \int P(x)dx$; note that R is a function of x. Hence we have a solution

$$\log y(x) = -R(x) + A' \qquad (10)$$

where A' is *a constant of integration.* Taking exponentials of both sides we rewrite as

$$y(x) = Ae^{-R(x)} \tag{11}$$

where A is the constant of integration in a disguised form.

§**Inhomogeneous equation using an integrating factor**

This case Eq. (8) can be solved by using (a) the solution of the homogeneous equation Eq. (7) as *an integrating factor*, and (b) convert to a simpler equation solvable by separation of variables.

We substitute into Eq. (8)

$$y(x) = e^{-R(x)}\phi(x) \tag{12}$$

where $\phi(x)$ is undetermined. We know that if $Q = 0$ then ϕ would be a constant, this is the real clue to what we are doing here. The factor $e^{-R(x)}$ in Eq. (12) is called *the integrating factor.*

Taking derivative of Eq. (12)

$$y' = -R'y + e^{-R(x)}\phi'(x),$$

and using $R' = P$ we see that $y' + Py = e^{-R(x)}\phi'(x)$.

Hence Eq. (8) becomes

$$\begin{aligned} e^{-R(x)}\phi'(x) &= Q(x) \\ \text{or } \phi'(x) &= e^{R(x)}Q(x), \end{aligned} \tag{13}$$

and hence the 'formal' solution is

$$\phi(x) = \int_0^x e^{R(x')}Q(x')\,dx' + A \tag{14}$$

where A is a constant of integration, and therefore the solution for y is

$$y(x) = e^{-R(x)}\left(\int_0^x e^{R(x')}Q(x')\,dx' + A\right) \tag{15}$$

This is the general solution.

It may help if we specialize to the case where the solutions required satisfy some initial condition: $y(0) = Y_0$ at $x = 0$. In this case we rewrite the above equation, changing to primed variables for clarity.

$$y(x) = e^{-R(x)} \left(\int_0^x e^{R(x')} Q(x')\, dx' + Y_0. \right), \qquad (16)$$

$$R(x) = \int_0^x P(x')\, dx'. \qquad (17)$$

Note that in this particular solution, we can easily verify $R(0) = 0$, and hence $y(0) = Y_0$. It is conventional to be somewhat vague in writing general solutions, but to be very precise when writing particular solutions.

§**First example:**

One example of inhomogeneous problem (in silent mode- i.e. with no commentary)

$$P = 2x, Q = \sin x,$$

$$y' + 2xy = \sin(x).$$

Solving homogeneous equation

$$R(x) = e^{-x^2}$$

Hence writing $y = e^{-x^2}\phi(x)$

$$\phi'(x) = e^{x^2} \sin x,$$

Hence the solution is:

$$y(x) = e^{-x^2} \left(y(0) + \int_0^x e^{x'^2} \sin x'\, dx' \right)$$

§**A falling mass in a viscous medium:**

MB. 400.26

A particle of mass m falls in a long tube filled with a viscous medium. It motion is described by modifying Newton's laws to include viscous slowing down:

$$m\frac{dv}{dt} = -mg - \eta v,$$

where $\eta > 0$ is a viscosity constant representing drag due to the medium. To see its effect, note that when $v < 0$ the viscous force is positive, i.e. upwards, due to the sign convention used. Similarly, if we threw the ball up, i.e. if $v > 0$, the drag force would point downward. Hence whatever we want to do is opposed by this drag force, hence the name.

Note that under changing the sign of both v and t, the equation is not invariant, unlike the problem with zero viscosity.

Let u s solve the problem using the method of an integrating factor.

We may analyze the motion under this equation and show that it has a terminal velocity.

Solution:

$$v(t) = v_0 e^{-\eta t/m} - \frac{mg}{\eta}(1 - e^{-\eta t/m})$$

§**Terminal velocity:**

$$v_{terminal} = -\frac{mg}{\eta}.$$

§**Bernoulli's trick:**

Some non-linear equations can be reduced to linear ones by a change of variables of the *dependent variable*. See Sec 4. in MB.

$$y' + P(x)y = Q(x)y^n,$$

AKA

$$dy + y\,P(x)\,dx = y^n\,Q(x)\,dx$$

can be linearized by changing variables $y \to z$

$$z = y^{1-n},$$

$$dy = dz\,y^n\,/\{(1-n)\}$$

We substitute and cancel y^n to find

$$dz/\{1-n\} + y^{1-n}P(x)\,dx = Q(x)\,dx$$

i.e.

$$z' + (1-n)P(x)z = (1-n)Q(x).$$

Rest is as seen before.

§Exact Equations: Integrating factors:

Suppose we are given an equation

$$\frac{dy}{dx} = -\frac{P(x,y)}{Q(x,y)}, \tag{18}$$

which can be rewritten as

$$Pdx + Qdy = 0.$$

In some cases, such an equation can be solved using a trick from partial differentials. Suppose we can find two functions of both x and y, called $F(x,y)$ and $R(x,y)$, with the property that

$$\begin{aligned} \frac{\partial F}{\partial x} &= P(x,y)\,e^{R(x,y)} \\ \frac{\partial F}{\partial y} &= Q(x,y)\,e^{R(x,y)}. \end{aligned} \tag{19}$$

There is no guarantee in general that we can succeed in this task, but let us assume we did succeed in finding such an F. The payoff is that we are now on track for solving our Eq. (18). Let us see how this happens. By the way $e^{R(x,y)}$ is called an integrating factor, and to start with, it is a good idea to first check if $R = 0$ works. If it fails, we can learn enough from the failed attempt to guess an R, as seen in the example below.

§Formal part of exact differentials:

To see this note that a variation of an given function of two variables $F(x,y)$, is given by

$$dF(x,y) = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy \tag{20}$$

with the property following from the general principle of partial derivatives (an no more), that

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x}. \tag{21}$$

Since we assumed success in finding an F with the property in Eq. (20) satisfied, we can say

$$e^{R(x,y)} \left(P(x,y)dx + Q(x,y)\, dy\right) = dF(x,y) \tag{22}$$

Now comes the checking part: if our guess of R and F is to be meaningful, we must make sure that the condition Eq. (22) is fulfilled. This means we must check if these equivalent conditions are fulfilled

$$\begin{aligned} \partial_y(e^{R(x,y)}P) &= \partial_x(e^{R(x,y)}Q) \\ R_y P + P_y &= R_x Q + Q_x \end{aligned} \tag{23}$$

where we used $\partial_y e^{R(x,y)} = e^{R(x,y)} R_y(x,y)$ etc, and canceled a factor of R. We can rewrite this as

$$(P_y - Q_x) = R_x Q - R_y P. \tag{24}$$

If it is fulfilled we have a solution, since the Eq. (18) is equivalent to

$$\begin{aligned} dF(x,y) &= 0 \\ F(x,y) &= \text{constant}. \end{aligned} \tag{25}$$

§**Simple example with $R = 0$:**

$$\frac{dy}{dx} = -\frac{y}{x}$$

or

$$ydx + xdy = 0$$

Thus $P = y$ and $Q = x$ and hence we find $P_y = 1 = Q_x$ and hence Eq. (24) is fulfilled with $R = 0$.

We still need to find F, we know that $\partial_x F = y$ and $\partial_y F = x$ and so we can integrate either of these to find:

$$F = xy + A,$$

where A is a constant.

§**Another problem: solved equivalently in MB:**

$$\frac{dy}{dx} = \frac{y}{x}$$

$$ydx - xdy = 0 \tag{26}$$

$P = y$ and $Q = -x$ here. Thus $P_y = 1 = -Q_x$ and hence we need an R. We now rewrite our Eq. (24) with the aim to find a possible R.

$$\begin{aligned} (P_y - Q_x) &= R_x Q - R_y P \\ \text{i.e. } 2 &= -R_x x - R_y y \end{aligned} \tag{27}$$

There is hope here. We can try to balance this equation by assuming R is a function of (say) x only. (The other choice is to make R a function of y only and gives the same answer at the end- we will check this.)

Since $R(x,y) \to R(x)$, $R_y =$ and Eq. (27) becomes

$$R_x = -\frac{2}{x}$$

i.e.

$$R = -2\log x$$

Therefore Eq. (26) is equivalent to Eq. (22) with

$$\begin{aligned} dF(x,y) &= \frac{-2}{x^2}(ydx - xdy) \\ &= 2\left(\frac{dy}{x} - \frac{y\,dx}{x^2}\right). \end{aligned} \tag{28}$$

We see that this corresponds to

$$F(x,y) = 2\frac{y}{x},$$

and hence the family of solutions to the problem Eq. (26) is

$$\frac{y}{x} = A.$$

If we had chosen the other possibility, i.e. R is a function of y alone, then we can grind through and conclude

$$\frac{x}{y} = B,$$

which is the same solution.

§**Another example of exact differentials:**

Problem MB 403.13
We want to solve

$$dx + (x - e^y)dy = 0, \quad \text{(Problem \# 2)}$$

with $P = 1$ and $Q = x - e^y$. Hence using our formalism, $P_y = 0$ and $Q_x = 1$ and hence we need to find R from Eq. (24), which now reads

$$-1 = R_x(x - e^y) - R_y.$$

We can choose R as a function of y alone to kill the first term and this gives

$$R = y$$

and hence we rewrite the given (Problem #2) as

$$dF(x, y) = e^y dx + (e^y x - e^{2y})dy = 0$$

and since this is a consistent exact differential,

$$F_x = e^y, \ldots (I) \;\; F_y = (e^y x - e^{2y}) \ldots (II)$$

Taking the first equation (I) we integrate it to find

$$F = xe^y + \phi(y),$$

where $\partial_x \phi(y) = 0$ and hence we are allowed a function of y, to play the role of a constant of integration of (I). We can now plug this into (II) and find

$$xe^y + \partial_y \phi(y) = xe^y - e^{2y}.$$

Canceling common terms we get an equation for ϕ as

$$\partial_y \phi(y) = -e^{2y},$$

and hence

$$\phi(y) = \frac{1}{2}e^{2y} + A$$

$$F = e^y x - \frac{1}{2}e^{2y} + A$$

where A is a constant that can be chosen as we wish.

Therefore we conclude that Problem #2 admits a family of solutions given by

$$e^y x - \frac{1}{2}e^{2y} = \text{constant}.$$