Physics 116B- Spring 2018

Mathematical Methods 116 B

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Notes on Differential Equations-III

§ **Summary of last two lectures: Linear first order equations and integrating factors and Exact differentials:**

a) We discussed linear first order ODE's

$$y' + P(x)y = Q(x) \tag{1}$$

where, as an example, $P = x^2$ and $Q = e^x$, and hence

$$y' + x^2 y = e^x.$$

We use the separation of variables and solve the homogeneous equation (i.e. setting $Q = 0$ temporarily)

$$\frac{dy}{y} = -x^2\,dx \tag{2}$$

which is solved by integration

$$y = Ae^{-x^3/3}.$$

and putting it in the form

$$y(x) = Ae^{-R(x)} \tag{3}$$

$R = -x^3/3$ is the integrating factor, used in the assumed functional form

$$y(x) = e^{-R(x)}\phi(x) \tag{4}$$

where $\phi(x)$ is determined by plugging into the differential equation Eq. (1), which now gives

$$\begin{aligned}\phi'(x) &= e^{R(x)}Q(x) \\ &= e^{x^3/3}e^x,\end{aligned} \tag{5}$$

and hence the 'formal' solution is

$$\phi(x) = \int_0^x e^{x'^3/3} e^{x'} \, dx' + A \tag{6}$$

where A is a constant of integration.

Writing back in the original variables we get

$$y(x) \quad = \quad e^{-x'^3/3} \left(\int_0^x e^{x'^3/3} e^{x'} \, dx' + Y_0. \right) \tag{7}$$

b) We also learned about exact differential equations, a special class of solvable ODE's.

$$\frac{dy}{dx} \quad = \quad -\frac{P(x,y)}{Q(x,y)}, \tag{8}$$

or, by cross-multiplication

$$P dx + Q dy \quad = 0.$$

The basic observation is that if we can find a function $F(x, y)$ in terms of which we can write

$$dF(x,y) \quad = \quad e^{R(x,y)} \, (P dx + Q dy) \tag{9}$$

then the problem is easily solved. This means we should be able to find the integrating factor e^R such that the condition

$$R_y P - R_x Q \quad = \quad Q_x - P_y$$

is fulfilled. Here R_y is the partial derivative wrt y etc. Remember that $R = 0$ is allowed. We are just looking for any R such that this condition is solved. We saw several examples of such equations.

§**Homogeneous Equations:**

We can solve rather complicated ODE's, if P and Q have the property of homogeneity with the same degree! Definition and examples of homogeneity:

$$A(hx, hy) = h^m A(x,y) \tag{10}$$

Here m is the degree of homogeneity. If this property is true then we can be sure that we can write

$$A(x,y) = x^m \alpha(v), \quad v = \frac{y}{x} \tag{11}$$

where α can be found from A easily.

Let us consider an example

$$A(x,y) = 3x^3 + 2x^2 y + 6xy^2 + 11y^3, \tag{12}$$

which clearly satisfies Eq. (10) with $m = 3$, and

$$\alpha(v) = 3 + 2v + 6v^2 + 11v^3. \tag{13}$$

Let us now go back to the equation

$$P(x,y)dx + Q(x,y)dy = 0, \tag{14}$$

with

$$\begin{aligned} P(tx,ty) &= t^m P(x,y), \quad P(x,y) = x^m \widetilde{P}(v) \\ Q(tx,ty) &= t^m Q(x,y), \quad Q(x,y) = x^m \widetilde{Q}(v), \end{aligned} \tag{15}$$

where

$$v = \frac{y}{x}.$$

Here comes the important point, we can rewrite Eq. (14) by canceling the common factor x^m, as

$$\widetilde{P}(v)dx + \widetilde{Q}(v)dy = 0. \tag{16}$$

we now trade the variable y in favor of v by writing $y = vx$. This implies

$$dy = vdx + xdv, \tag{17}$$

and plugging into Eq. (16) we get an equation in terms of x, v as

$$(v\widetilde{Q}(v) + \widetilde{P}(v))dx + x\widetilde{Q}(v)dv = 0. \tag{18}$$

This is separable and can be written as

$$\frac{dx}{x} + \frac{\widetilde{Q}(v)}{(v\widetilde{Q}(v) + \widetilde{P}(v))}\, dv = 0. \tag{19}$$

We can simplify the notation by writing a new function

$$f(v) = \frac{\widetilde{Q}(v)}{(v\widetilde{Q}(v) + \widetilde{P}(v))} \tag{20}$$

so that the Eq. (19) becomes

$$\frac{dx}{x} + f(v)\, dv = 0. \tag{21}$$

We have learnt how to solve this by integration, we get the solution as

$$x = x_0 e^{-\int_c^v dv' f(v')} \tag{22}$$

where x_0, c are initial values to be fixed later ($x = x_0$ when $v = c$), and we should convert back to y by using $v = y/x$.

§**An example:**

$$xydx + (y^2 - x^2)dy = 0. \tag{23}$$

Clearly $P = xy$ and $Q = y^2 - x^2$ are homogeneous of degree $m = 2$.

With $y = vx$,

$$\widetilde{P}(v) = v, \;\; \widetilde{Q}(v) = v^2 - 1,$$

and

$$f(v) = \frac{v^2 - 1}{v(v^2 - 1) + v} = \frac{1}{v} - \frac{1}{v^3}$$

Hence

$$\int_c^v f(v) = \log(v/c) + \frac{1}{2}(\frac{1}{v^2} - \frac{1}{c^2})$$

where c is some initial value of v. Notice that in this case $c = 0$ would be a *bad choice.*

Solution in terms of x, y is found by plugging in $v = y/x$ into Eq. (22). We get

$$y = cx_0 e^{-\frac{1}{2}(\frac{x^2}{y^2} - \frac{1}{c^2})} \tag{24}$$

At this point, we can also club together the three constant factors and rewrite this more conveniently as

$$y = Ae^{-\frac{1}{2}(\frac{x^2}{y^2})}, \tag{25}$$

and we are at freedom to choose the initial condition, now at $x = 0$ we can set $y = y_0$ (and hence $A = y_0$).

§Linear equations and Superposition of solutions versus Nonlinear equations:

For linear "operators", i.e. for linear equations written symbolically

$$\begin{aligned} Ly_1 &= 0 \\ Ly_2 &= 0 \\ Ly_3 &= 0 \\ \ldots &= 0 \end{aligned} \tag{26}$$

This implies

$$L(c_1y_1 + c_2y_2 + c_3y_3 + \ldots) = 0,$$

i.e. we can add solutions as we choose. This is superposition of solutions and of crucial importance in Quantum Theory, where the Schrodeinger equation is linear!

Not so for non-linear equations. Cannot add solutions, cannot even multiply solutions with constants.

§Riccati equation:

This is a remarkable non-linear equation, which can often be solved.

The generalized Riccati equation is given by

$$y'(x) = f(x)y^2 + g(x)y + h(x) \tag{27}$$

where f, g, h are functions of x.

Let us keep an example in mind

$$y' = e^{-x}y^2 + y - e^x.$$

Hence $f = e^{-x}$, $g = 1$, $h = -e^x$.

We need some starter information to solve this problem. Somehow (mostly by inspection) if we have on particular solution, called $y_p(x)$ we can get the general solution. For the above special case, we see that $y_p = e^x$ solves it. It has unfortunately no additional parameters, so we cannot do much with it. (Note that since this is no-linear, even a constant multiple of e^x fails to solve the equation.) But the trick of Riccati overcomes this.

Plug in

$$y = y_p + u(x)$$

so that the equation reduces to a Bernoulli equation

$$u' = (2y_p f(x) + g(x))u(x) + f(x)u^2(x),$$

which can be simplified to a linear equation by using Bernoulli's trick. You will recall that we can linearize certain non-linear ODE's by redefining the dependent variable suitably. Here it means setting $u = 1/z$ and gives

$$z'(x) + z(2y_p f + g) + f(x) = 0.$$

This is linear and can be solved as usual. In our example, the equation reduces to

$$z'(x) + z(2e^x e^{-x} + 1) + e^{-x} = 0,$$

or

$$z'(x) + 3z + e^{-x} = 0.$$

Simple enough!!

§ **Linear differential equations with constant coefficients:**

Simple but important class of ODE's.

Let us define

$$D = \frac{d}{dx}, \quad D^2 = \frac{d^2}{dx^2}, \quad D^m = \frac{d^m}{dx^m}.$$

so we can write the various ODE's in terms of D in shorthand. We are interested in ODEs that look like

$$c_m D^m y + c_{m-1} D^{m-1} y + \dots c_1 D y = H(x) \tag{28}$$

where c_j are constants, i.e. independent of x. $H(x)$ on the right is the inhomogeneous term, which we set to zero initially. So we will first study the homogeneous problem

$$c_m D^m y + c_{m-1} D^{m-1} y + \dots c_1 D y = 0. \tag{29}$$

We can safely think of the $D^n \leftrightarrow d^n$ where d is a number (not a derivative) because c_j are constants and every term "commutes" with others.

Note that we can factorize a polynomial in d's into its roots.

$$c_m d^m + c_{m-1} d^{m-1} + + \dots c_1 d = c_m (d - a_1)(d - a_2) \dots (d - a_m) \tag{30}$$

We can divide out by c_m and hence shall set it to 1 below. Here the polynomial is written in terms of its (zeros) roots a_j. The usual methods can be used to find the zeros of the polynomials, in terms of the c_j's.

$$\{c_j\} \rightarrow \{a_j\}.$$

Since c_m are all real, a_m will be either all real or occur in complex conjugate pairs.

§**Simple solutions:**

§m=1

$$(D - a)y = 0$$

is solved by

$$y = y_0 e^{ax}.$$

§m=2

$$(D - a)(D - b)y = 0$$

Let us imagine a and b are unequal numbers- real or complex. One solution is obvious, if we set

$$(D - b)y = 0,$$

the equation is satisfied. This means *one solution* has been found, it is

$$y = Be^{bx}.$$

The hunt moves on, we want to find all solutions, so there is one more needed. We can solve for this equally easily by noting that the independence of a, b on x implies the two terms are interchangeable. Hence we could have written the equation as

$$(D - b)(D - a)y = 0,$$

and hence another solution is from the linear equation

$$(D - a)y = 0$$

i.e.

$$y = Ae^{ax}.$$

Combining, we write the general solution

$$y = Ae^{ax} + Be^{bx},$$

with two arbitrary parameters A, B.

§m=3

$$(D - a)(D - b)(D - c)y = 0$$

Do we need to calculate, or an we see the pattern above and write down the exact answer directly?

§**Complex roots:**

Going back to $m = 2$ case: if $a = b^*$ we can write the cartesian representation of them as

$$a = \alpha + i\beta, \quad b = \alpha - i\beta$$

The general *real* solution can be written in the convenient form

$$y = Ce^{\alpha x} \sin(\beta x + \phi).$$

This can be generalized easily to any number of complex roots. For every conjugate pair we extract a α, β and write the same expression, and sum over all roots.