Physics 116B- Spring 2018

Mathematical Methods 116 B

S. Shastry, April 26, 2018
Notes on Differential Equations-IV

§ **Summary of last lecture: Linear differential equations with constant coefficients:**

Simple but important class of ODE's.

$$D = \frac{d}{dx}, \quad D^2 = \frac{d^2}{dx^2}, \quad D^m = \frac{d^m}{dx^m}.$$

The linear ODE with constant coefficients c_j are written as

$$c_m D^m y + c_{m-1} D^{m-1} y + \ldots c_1 D y = 0 \tag{1}$$

and we also want to study the equations with $H(x)$ on the right hand side. We set $c_m = 1$ so that we can factorize a polynomial in d's into its roots.

$$(D - a_1)(D - a_2) \ldots (D - a_m) y = 0 \tag{2}$$

Here the polynomial is written in terms of its (zeros) roots a_j.

We saw for m=1

$$(D - a)y = 0$$

Solution is

$$y = y_0 e^{ax}.$$

m=2

$$(D - a)(D - b)y = 0$$

The general solution

$$y = Ae^{ax} + Be^{bx},$$

with two arbitrary parameters A, B.

§m=3 by analogy.

$$(D-a)(D-b)(D-c)y=0$$

We also discussed complex roots.
§End of summary. ——————————-

§Coincident (or double) roots:

What happens when $a=b$? Although one root is clearly found, the other poses an interesting issue. Let us go back to $m=2$ and write

$$(D-a)(D-a)y=0, \tag{3}$$

and one solution is

$$y=Ae^{ax} \tag{4}$$

The other root, as per Boas's book is written down as

$$y=Bxe^{ax}. \tag{5}$$

Let us verify that this is true. Set $B=1$ for convenience

$$\begin{aligned}(D-a)(D-a)xe^{ax} &= (D-a)\left(-axe^{ax}+axe^{ax}+e^{ax}\right)\\ &= (D-a)e^{ax}\\ &= 0.\end{aligned} \tag{6}$$

We also need to verify the linear independence of Eq. (4) and Eq. (5). Calculate the Wronskian. Recall

$$W(f,g)=fg'-f'g$$

and if $W\neq 0$ then f,g are linearly independent.

Applying it here:

$$W(Ae^{ax},Bxe^{ax})=ABe^{2ax}\left(ax+1-ax\right)=ABe^{2ax}\neq 0$$

hence the two solutions are linearly independent.

§ **A more systematic way to treat coincident roots:**

A more systematic way to handle Eq. (3) is to do a perturbation around the degeneracy, and consider the operator

$$\begin{aligned} L_\eta &\equiv (D-a-\eta)(D-a+\eta) = L_0 - \eta^2 && (7) \\ L_0 &= (D-a)^2. && (8) \end{aligned}$$

Here η is an arbitrary number, finally set to zero. We know that the solution of L_η is

$$\begin{aligned} L_\eta \Phi(x,\eta) &= 0 && (9) \\ \Phi(x,\eta) &= Ae^{(a+\eta)x} + Be^{(a-\eta)x} && (10) \end{aligned}$$

We can Taylor expand Eq. (9) around $\eta = 0$. Let

$$\Phi(x,\eta) = \Phi(x,0) + \eta\Phi'(x,0) + \frac{1}{2}\eta^2\Phi''(x,0) + \ldots.$$

Here we should keep in mind that $\Phi(x,\eta)$ is a function of *two variables*, x and η, and the derivatives in the above expansion are with respect to η (at a fixed x).

The coefficients are

$$\begin{aligned} \Phi(x,0) &= (A+B)e^{ax} \\ \Phi'(x,0) &= (A-B)xe^{ax} \\ \ldots &= \ldots && (11) \end{aligned}$$

Hence a Taylor expansion in powers of η (at a fixed x) reads

$$(L_0 - \eta^2)(\Phi(x,0) + \eta\Phi'(x,0) + \frac{1}{2}\eta^2\Phi''(x,0) + \ldots) = 0 \qquad (12)$$

Since η is arbitrary, each power of η in Eq. (15) *must vanish independently*! This is the key point. Let us expand out

$$\begin{aligned} \mathcal{O}(\eta^0): \; L_0\Phi(x,0) &= 0 && (13) \\ \mathcal{O}(\eta^1): \; L_0\Phi'(x,0) &= 0 && (14) \\ \mathcal{O}(\eta^2): \; \frac{1}{2}L_0 \; \Phi"(x,0) - \Phi(x,0) &= 0 && (15) \\ ... &= ... \end{aligned}$$

We can use Eq. (11) to substitute into this equation.

Note that Eq. (13) is obviously true. Now Eq. (14) says there is another solution for the same $L_0 y = 0$, namely

$$y = \Phi'(x,0) = (A - B)xe^{ax}, \tag{16}$$

where the prefactor of $A-B$ is arbitrary, given the linearity of the equations. Hence we have shown that

$$(D-a)^2(xe^{ax}) = 0$$

this agrees with the result quoted by Boas, but gives us a systematic way of treating degenerate roots.

§**Harmonic oscillator revisited:**

This important problem can be treated by our formulas easily.

Let y be the displacement of an oscillator

$$\begin{aligned} \ddot{y} &= -\omega^2 y, \quad \text{or} \\ (D^2 + \omega^2)y &= 0 \end{aligned} \tag{17}$$

where $D = \partial/\partial t$. We factorize and write

$$(D + i\omega)(D - i\omega)y = 0. \tag{18}$$

Solution follows from our previous analysis. Two equivalent forms of the solution are:

$$y = Ae^{i\omega t} + Be^{-i\omega t} = C\sin(\omega t + \phi).$$

§**Damped Harmonic oscillator:**

This is an example with a drag, or frictional force in addition to the spring. Let the displacement be denoted by y again,

$$m\frac{d^2y}{dt^2} = -ky - l\frac{dy}{dt},$$

where $l > 0$ is like the viscous drag in earlier examples. The sign of the drag term implies that it is directed opposite to that of the velocity, and hence resists acceleration. We divide out by m and rewrite it, as in Boas,

$$\frac{d^2y}{dt^2} = -\omega^2 y - 2b\frac{dy}{dt}, \tag{19}$$

where

$$b = \frac{l}{2m}.$$

Thus the ODE can be cast in the form

$$(D^2 + 2bD + \omega^2)y = 0. \tag{20}$$

Now the roots of the auxiliary polynomial $(D \to d)$

$$d^2 + 2bd + \omega^2 = (d - \lambda_1)(d - \lambda_2),$$

where

$$\lambda_1 = -b + \sqrt{b^2 - \omega^2} \tag{21}$$
$$\lambda_2 = -b - \sqrt{b^2 - \omega^2}. \tag{22}$$

We will call the radical

$$R = b^2 - \omega^2 \tag{23}$$

The solutions are clearly

$$y = Ae^{\lambda_1 t} + Be^{\lambda_2 t}.$$

We look at the roots more closely and distinguish between three cases depending on R

- (1) Underdamped i.e. oscillatory: $R < 0$

  This implies damped oscillations

$$\begin{aligned} \lambda_1 &= -b + i\Omega \\ \lambda_2 &= -b - i\Omega \\ \Omega &= \sqrt{\omega^2 - b^2} \end{aligned} \tag{24}$$

  Clearly $\Omega = \sqrt{-R}$.

  Hence the solution is

$$y = e^{-bt} A \sin(\Omega t + \phi),$$

- (2) Critically damped $R = 0$

  By taking the case of double (or coincident) roots we can see that the solution is

  $$y = e^{-bt}(A + tB)$$

- (3) Overdamped $R > 0$

  Here both roots λ_j are real and positive as long as $b > 0$. Hence the general solution is

  $$y = Ae^{\lambda_1 t} + Be^{\lambda_2 t}.$$