Physics 116B- Spring 2018

Mathematical Methods 116 B

S. Shastry, May 8, 10, 2018
Notes on Differential Equations-V

§ **Summary of last lecture: Linear differential equations with constant coefficients:**

§**Damped Harmonic oscillator:**

This is an example with a drag, or frictional force in addition to the spring.

$$\frac{d^2y}{dt^2} = -\omega^2 y - 2b\frac{dy}{dt}. \tag{1}$$

The ODE can be cast in the form

$$(D^2 + 2bD + \omega^2)y = 0. \tag{2}$$

Now the roots of the auxiliary polynomial ($D \to d$)

$$d^2 + 2bd + \omega^2 = (d - \lambda_1)(d - \lambda_2),$$

where

$$\lambda_1 = -b + \sqrt{b^2 - \omega^2} \tag{3}$$
$$\lambda_2 = -b - \sqrt{b^2 - \omega^2}. \tag{4}$$

The solutions are:

$$y = Ae^{\lambda_1 t} + Be^{\lambda_2 t}. \tag{5}$$

Due to damping, the solutions die away at long times provided $b \neq 0$. For $b = 0$ the undamped case, the solution oscillates at all later times as

$$y = y_0 \sin(\omega t + \phi).$$

§ **Driven i.e. forced damped oscillator:**

$$\frac{d^2y}{dt^2} + \omega^2 y + 2b\frac{dy}{dt} = F\sin(\omega' t) \tag{6}$$

where $\omega' \neq \omega$.

Linear ODE with constant coefficients and a non-zero RHS.

We could take the previous solution and add something to it:

$$y = Ae^{\lambda_1 t} + Be^{\lambda_2 t} + y_p,$$

this is plugged in to give

$$\frac{d^2y_p}{dt^2} + \omega^2 y_p + 2b\frac{dy_p}{dt} = F\sin(\omega' t) \tag{7}$$

If we can find *any single solution* of this equation, we are done since A and B have given two constants already, so we will have the most general solution.

$$y_p = \text{Particular solution.}$$

We now make it simpler by making it complex!!

$$\sin(\omega' t) = \Im m\, e^{i\omega' t} \tag{8}$$

We will solve

$$\frac{d^2Y_p}{dt^2} + \omega^2 Y_p + 2b\frac{dY_p}{dt} = Fe^{i\omega' t} \tag{9}$$

and take imaginary part of both sides of the equation. This will guarantee that

$$\Im m\, Y_p = y_p.$$

Guess for solution in complex domain.....?????

Try

$$Y_p = Ce^{i\omega' t}$$

Derivatives:

$$D^n Y_p = (i\omega')^n Y_p$$

Hence

$$(\omega^2 - \omega'^2 + 2ib\omega')C = F$$

$$C = \frac{F}{(\omega^2 - \omega'^2) + 2ib\omega'} = \frac{Fe^{-i\phi}}{\sqrt{(\omega^2 - \omega'^2)^2 + 4b^2\omega'^2}}$$

ϕ is a phase shift.

Hence

$$y_p = \Im m\,(Y_p) = \frac{F \sin(\omega' t - \phi)}{\sqrt{(\omega^2 - \omega'^2)^2 + 4b^2\omega'^2}}$$

Tuning the TV is to adjust ω to match ω'.

§ **Other examples of non-zero RHS:**

Section 6 Chapter 8..
Example problem:

$$(D-2)(D-5)y = \sin(3t)$$

Try

$$y = Ae^{2t} + Be^{5t} + y_p,$$

where

$y_p =$ Any old solution, no matter how simple, of given equation

One way:

Complexify and put

$$y_p = Y_p = Ce^{i3t}$$

satisfying

$$(D-2)(D-5)Y_p = e^{i3t}$$

or

$$C(i3-2)(i3-5) = 1$$

and hence (pl check this)

$$y_p = -\sin(3t - \arctan(21))/\sqrt{442}.$$

Another method to find the particular solution: Successive integration: We can call

$$(D-2)y = W(t),$$

so the equation for W is

$$dW/dt - 5W = \sin(3t).$$

Linear equation with constant term on right. Use integrating factor

$$W = e^{5t} \int dt e^{-5t} \sin(3t) + Ae^{5t}$$

or

$$W(t) = -\frac{1}{34} e^{-5t}(3\cos(3t) + 5\sin(3t)) + Ae^{5t}$$

we can now get back to y using

$$dy/dt - 2y = W(t)$$

Solution of this is straightforward but tedious:

$$y(t) = \left\{ \frac{442Ae^{5t} + 3\sin(3t) + 63\cos(3t)}{1326} + c_1 e^{2t} \right\}$$

Question: Is this y or y_p?

§**Exponential RHS:**

If we have an equation

$$(D-a)(D-b)y = ke^{cx} \tag{10}$$

we can solve this much more easily. Let $c \neq a$ and $c \neq b$ for now.

Note that

$$De^{cx} = ce^{cx}$$

$$D^2e^{cx} = c^2e^{cx}$$

etc

We can now get more adventurous,and justify rigor later...

Let us try to invert:

$$(D-a)e^{cx} = (c-a)e^{cx}$$

i.e.

$$e^{cx} =??\frac{(c-a)}{(D-a)}e^{cx}$$

Check this by series expansion of denominator

Yes...

Hence as long as we do not hit a zero in the denominator we can invert:

i.e. to be more precise

$$(D-a)y = ke^{cx} \implies y = k\frac{1}{D-a}e^{cx} = k\frac{1}{c-a}e^{cx}$$

provided $c \neq a$.

Hence the solution of Eq(10) is obvious now

$$y_p = \frac{k}{(c-a)(c-b)}e^{cx}$$

if $c \neq a$ and $c \neq b$.

Caution

$$(D-a)e^{cx} \neq e^{cx}(D-a)$$

Ordering is important in handling the derivative operators.

This is the first example of *non-commutative objects* in Quantum theory, where $p \to -i\hbar\partial_x$.

§What about coincident c=a?:

We want to solve

$$(D-a)(D-b)y = ke^{ax} \tag{11}$$

so we cannot be too cavalier!!

Any ideas?

Think perturbation...

Put $c = a + \epsilon$ where ϵ is small. As long as it is non-zero we have the old solution for $c \neq a, c \neq b$.

$$(D-a)(D-b)y = ke^{(a+\epsilon)x} \implies y = \frac{k}{(a+\epsilon-b)\epsilon}\ e^{(a+\epsilon)x} \tag{12}$$

Firstly recall we are looking for particular solutions, which means we can drop any additional things like αe^{ax} in

$$y = y_p + \alpha e^{ax}$$

using the property that

$$(D-a)(D-b)\alpha e^{ax} = 0 \tag{13}$$

where α is arbitrary.

Now let us Taylor expand:

$$e^{(a+\epsilon)x} = e^{ax}(1 + \epsilon x + O(\epsilon^2))$$

Hence

$$\begin{aligned} y_p &= \frac{k}{(a+\epsilon-b)\epsilon}\; e^{ax}(1+\epsilon x+O(\epsilon^2)) \\ &\to \frac{k}{(a-b)}\; xe^{ax} \end{aligned} \quad (14)$$

Summarizing:

$$(D-a)(D-b)y_p = ke^{ax} \implies y = \frac{k}{(a-b)}\; xe^{ax} \quad (15)$$

Similarly also

$$(D-a)(D-a)y_p = ke^{ax} \implies y = 2k\; x^2e^{ax} \quad (16)$$

§ **Theorem:**

$$(D-a)(D-b)y = e^{cx}P_n(x) \quad (17)$$

where P_n is a polynomial of degree n

$$y_p = e^{cx}Q_n(x) \quad (18)$$

where Q_n is a polynomial of *same degree*. It can be found by plug-n-play!!

Other special cases/examples of solvable second order differential equations.

§**Newtonian equations of motion: Equations independent of the independent variable!!:**

Suppose we are given

$$y'' + \Phi(y) = 0 \tag{19}$$

where $y' = dy/dx$, and $\Phi(y)$ is some function of y only, and not of x explicitly. Here x is the independent variable and we note that the Eq. (19) is independent of x!!

We recognize this as Newtons laws by rewriting as

$$\begin{aligned} m\ddot{x} &= -\frac{\partial V(x)}{\partial x} && (20)\\ E_{tot} &= \frac{1}{2}m(\dot{x})^2 + V(x) && (21) \end{aligned}$$

where V is the potential energy and its derivative is the force. Eq. (21) is the energy of the particle which is independent of x, i.e. the particle moves in such a way that the kinetic and potential energies balance out their x dependencies to give a constant energy. We can map Eq. (20) to the first equation by dividing through with m and renaming objects. Let us do the renaming in clear terms: *alert: most confusions arise from forgetting the details of this relabeling*

$$y \to x, \;\; x \to t, \;\; \Phi(y) \to \frac{1}{m}\frac{\partial V(x)}{\partial x}.$$

So the thing to note is that Eq. (19) is a second order differential equation where x is *implicit* and not explicit. We now erase the comment made above, and try to solve the problem. We should of course discover all the good things said above.

To solve Eq. (20) multiply both sides by y', thus

$$y'y'' + y'\Phi(y) = 0 \tag{22}$$

Now

$$y'y'' = \frac{1}{2}\frac{d(y'^2)}{dx},$$

and

$$y'\Phi(y) = \frac{d}{dx}\int_0^y \Phi(s)\,ds$$

We may thus write Eq. (22) as

$$\frac{d}{dx}\left(\frac{1}{2}(y')^2 + \int_0^y \Phi(s)\,ds\right) = 0, \tag{23}$$

Now we relabel $\int_0^y \Phi(s)\,ds = mV(y)$ therefore

$$\frac{1}{2}(y')^2 + mV(y) = \text{constant}. \tag{24}$$

Constant = ??

$$\frac{1}{2}(y')^2 + mV(y) = mE. \tag{25}$$

Energy

This almost solves the differential equation Eq. (19), and we have discovered the principle of energy conservation! We still need to solve it fully. For this let us work out an example:

Example:

$$\Phi(y) = -y + 2y^3$$

Eqn:

$$y'' - y + 2y^3 = 0$$

Observe no dependence on x

Step 1. Multiply by y'

$$y'y'' + y'(-y + 2y^3) = 0$$

Becomes

$$\frac{d}{dx}\left(\frac{1}{2}(y')^2 - \frac{1}{2}y^2 + \frac{1}{2}y^4\right) = 0$$

hence we have shown that

$$\left(\frac{1}{2}(y')^2 - \frac{1}{2}y^2 + \frac{1}{2}y^4\right) = \frac{1}{2}E$$

where E is a constant. We know by now, that it is the Newtonian energy of the particle, but we need not talk about the physical meaning as yet.

To get the full solution we need one more step.

Step 2. By transferring to one side we isolate the velocity y'

$$y' = \pm\sqrt{E + y^2 - y^4}$$

We can separate variables as

$$\frac{dy}{\sqrt{E + y^2 - y^4}} = dx$$

This can be integrated, see MMA notebook for answer. It is a bit complicated and involves elliptic functions, so we skip it here.

Let us take a more "physical" way to understand the behavior without solving it explicitly. Firstly let us solve for the turning points. These are the maximum or least value of y at which the velocity is real are given by

$$y = \pm y^*(E), \quad \text{where} \quad y^*(E) = \sqrt{\frac{1}{2} + \sqrt{E + \frac{1}{4}}}$$

There are two possible signs and both are allowed.

Physically we may describe the motion of the particle, now for convenience think of y as the position of the particle and the independent variable as time. See discussion on mathematica page on website.

Anharmonic oscillator differential equation solution. May 10, 2018

Our problem is to solve

$y'' - y + 2\, y^3 = 0$

This corresponds to *a* Newtonian equation of motion

$1/2\,(y')^2 + V(y) = E$

$V(y) = 1/2\left(-y^2 + y^4\right)$

See Eq. 25 in class notes of May 8, 10

The integral of the quartic is a well known elliptic function. We do not study it in this course, but it is a very well known function at an advanced level.

In[1]:=
```
Integrate[1 / Sqrt[A / 4 + y^2 - y^4], y]
```

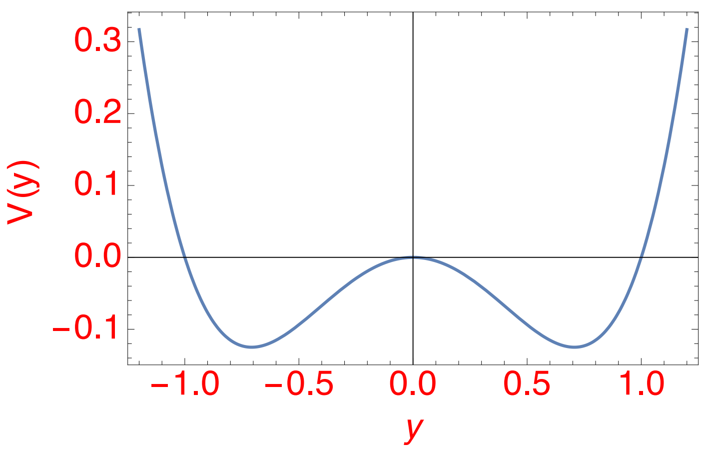
Out[1]=

$$-\left(\left(\mathrm{i}\sqrt{2}\sqrt{1-\frac{2\,y^2}{1-\sqrt{1+A}}}\sqrt{1-\frac{2\,y^2}{1+\sqrt{1+A}}}\ \text{EllipticF}\left[\mathrm{i}\,\text{ArcSinh}\left[\sqrt{2}\sqrt{-\frac{1}{1-\sqrt{1+A}}}\,y\right],\frac{1-\sqrt{1+A}}{1+\sqrt{1+A}}\right]\right)\Bigg/\left(\sqrt{-\frac{1}{1-\sqrt{1+A}}}\sqrt{A+4\,y^2-4\,y^4}\right)\right)$$

To better understand these let us plot the potential V(y)= -y^2/2+y^4/2

In[2]:=
```
Plot[-y^2 / 2 + y^4 / 2, {y, -1.2, 1.2}, LabelStyle → Red,
 BaseStyle → {FontSize → 18, FontFamily → "Helvetica"} ,
 Frame → True, FrameLabel → {y, "V(y)"}, PlotRange -> All]
```

Out[2]=



The function u is the radical, i.e. y'= +₋ $u^{1/2}$(y,E). I wrote A in place of E,because Mathematica is fussy about the symbol E, which is "reserved".

In[3]:=
```
u[y_, A_] = Sqrt[A + y^2 - y^4];
```

For a given A, the largest value of y that keeps u real is given by g(A) which is given next.

In[4]:=
```
g[A_] = Sqrt[1 / 2 + Sqrt[A + 1 / 4]]
```

Out[4]=

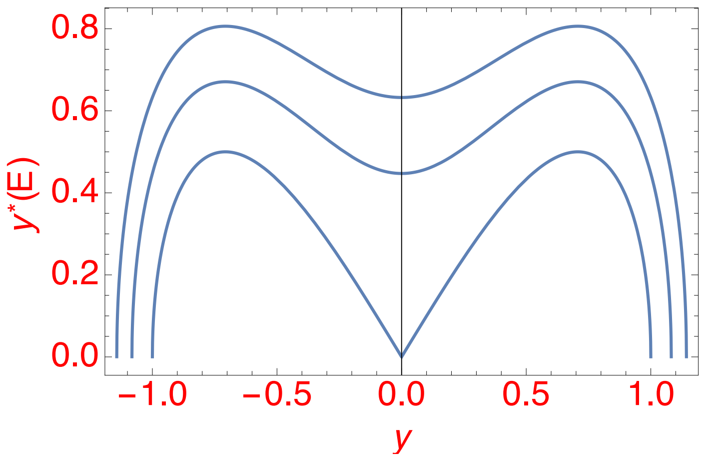
$$\sqrt{\frac{1}{2}+\sqrt{\frac{1}{4}+A}}$$

In the 4 figures below we plot these s

In[5]:=
```
pl[A_] := Plot[u[y, A], {y, -g[A], g[A]}, PlotRange -> All]
```

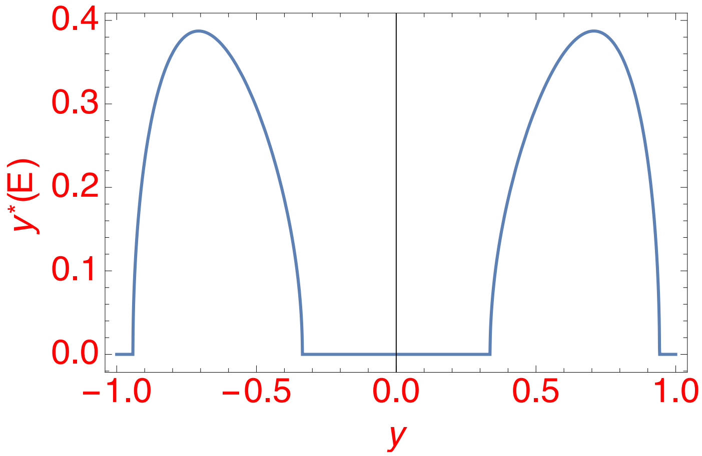
In[6]:=
```
Show[{pl[.4], pl[.2], pl[0]}, LabelStyle → Red,
 BaseStyle → {FontSize → 18, FontFamily → "Helvetica"},
 Frame → True, FrameLabel → {y, "y* (E)"}, PlotRange -> All]
```

Out[6]=



In[7]:=
```
Plot[Re[u[y, -.1]], {y, -1., 1.}, LabelStyle → Red,
 BaseStyle → {FontSize → 18, FontFamily → "Helvetica"},
 Frame → True, FrameLabel → {y, "y* (E)"}, PlotRange -> All]
```

Out[7]=



§ **Second order equation independent of y (but depends on y', y'' etc.):**

As an example consider

$$y'' + 3y' = f(x) \tag{26}$$

and we will encounter other problems of this type in the HW#5. We can solve this problem for any $f(x)$ by a simple trick, which exploits the absence of y in the equation.

Let us call

$$p(x) = y'(x)$$

so that $p' = y''$. Hence the equation simplifies to a first order equation for p.

$$p' + 3p = f(x) \tag{27}$$

We can solve this using the integrating factor trick for first order equations.

$$p(x) = e^{-3x} \int e^{3x'} f(x')\, dx' + Ae^{-3x} \tag{28}$$

where A is arbitrary.

We can now back up and find y by using

$$dy/dx = e^{-3x} \int e^{3x'} f(x')\, dx' + Ae^{-3x} \tag{29}$$

which is separable (note that y appears only on the left).

§**Change of independent variable: Cauchy-Euler equations:**

In equations of the type

$$x^2 \frac{d^2}{dx^2} y + bx \frac{d}{dx} y + c = d \tag{30}$$

where b,c,d are independent of x (they could depend on y!!) we use a simple trick.

Call

$$x = e^t, \text{ hence } x\frac{d}{dx} = \frac{d}{dt}.$$

Hence

$$\frac{d^2}{dt^2}y + b\frac{d}{dt}y + c = d \tag{31}$$

this can be solved more easily, because there are no longer the explicit troublesome factors of x, x^2.

§**Laplace Transforms and their application to solving ODE with constant coefficients:**

What is the Laplace transform?
If we are given

$$f(t), \quad t \in [0, \infty)$$

we can define its Laplace transform $L(f(t))$ as $F(p)$ where

$$F(p) \leftrightarrow f(t) \tag{32}$$

where $p > 0$. The relationship is more precisely

$$L(f(t)) : \quad F(p) = \int_0^\infty f(t) e^{-pt}\, dt \tag{33}$$

Inverting this to find $F(p)$ from $f(t)$ is a bit more complicated than in the case of a similar Fourier transform. We will bypass this by consulting a set of standard Laplace transforms if we need to invert, and perhaps after doing complex integration, we can revisit this.

Let us take a few examples of Laplace transforms
(a)

$$f(t) = 1, \implies F(p) = \frac{1}{p}$$

(b)

$$f(t) = t^n, \implies F(p) = \int_0^\infty t^n e^{-pt} dt \; = (-1)^n \frac{d}{dp}\frac{1}{p}$$

Thus

$$f(t) = t^n, \implies F(p) = \Gamma(1+n)\frac{1}{p^{1+n}}$$

(c)

$$f(t) = e^{-at}, \implies F(p) = \frac{1}{p+a}$$

This is true as long as $\Re e\,(p+a) > 0$, and includes the case when $a = i\alpha$ with real α.

§**A short break from calculations:**

Let us see what the Laplace transform is doing. It is simply an integration of the given function $f(t)$ after multiplying by e^{-st}. Hence the following (linearity) properties are obvious -

$$f(t) = c_1 f_1(t) + \ldots + c_n f_n(t), \implies F(p) = c_1 F_1(p) + \ldots + c_n F_n(p) \quad (34)$$

§ **Back to calculations of Laplace Transforms:**

Let us use this right away.
Want the Laplace transform of $\sin(\omega t)$ and we are given (c), i.e.

$$f(t) = e^{-at}, \implies F(p) = \frac{1}{p+a}$$

Now write

$$\sin(\omega t) = \frac{1}{2i} e^{i\omega t} - \frac{1}{2i} e^{-i\omega t}$$

Therefore

$$\begin{aligned} L(\sin(\omega t)) : \quad F(p) &= \frac{1}{2i}\frac{1}{p - i\omega} - \frac{1}{2i}\frac{1}{p + i\omega} \\ &= \frac{\omega}{p^2 + \omega^2} \end{aligned} \quad (35)$$

§**Using Laplace transform to solve ODE with constant coefficients:**

Idea is simple:
a) Start with ODE- Take Laplace transform of both sides
b) Now equation is algebraic and not an ODE at all, thanks to Laplace
c)Solve algebraic equation
d) To get back required solution of ODE, find the inverse Laplace transform of algebraic solution.

Take a truly simple example:

$$y' + 3y = \sin(\omega t) \tag{36}$$

Let

$$L(y): \quad Y(p) = \int_0^\infty e^{-pt} y(t)\, dt$$

Multiply Eq. (36) by $e^{-pt}\, dt$ and integrate. Hence

$$\int_0^\infty e^{-pt} y'(t)\, dt + 3Y(p) = \frac{\omega}{p^2+\omega^2} \tag{37}$$

where we used our recent example to write down the RHS i.e. $L(\sin)$.

For the first term we use integration by parts:

$$\int_0^\infty e^{-pt} y'(t)\, dt = [e^{-pt} y(t)]_0^\infty + p\int_0^\infty e^{-pt} y(t)\, dt = pY(p) - y(0).$$

Notice that the initial condition of y has made an explicit appearance here.

Hence the algebraic equation resulting from Laplacing the original equation is now:

$$\begin{aligned} (3+p)Y(p) &= y(0) + \frac{\omega}{p^2+\omega^2} \\ Y(p) &= \frac{y(0)}{3+p} + \frac{1}{3+p} \times \frac{\omega}{p^2+\omega^2} \end{aligned} \tag{38}$$

The final step is to take the inverse Laplace transform. The first term is easy: $y(0)e^{-3t}$.

Second term can be done by looking up tables, or best by using partial fractions.