Physics 116B- Spring 2018

Mathematical Methods 116 B

S. Shastry, May 15, 10, 2018
Notes on Differential Equations-VI

§ **Summary of last item in the last lecture::**

§**Laplace Transforms and their application to solving ODE with constant coefficients:**

With $p > 0$ the Laplace transform is defined as

$$L(f(t)) : \quad F(p) = \int_0^\infty f(t) e^{-pt} \, dt \tag{1}$$

We will consult a set of standard Laplace transforms if we need to invert. In the HW and the exams you will need a copy of the Laplace transform tables given in the book.

For solving ODEs with constant coefficients, the basic idea is simple:
a) Start with ODE- Take Laplace transform of both sides
b) Now equation is algebraic and not an ODE at all, thanks to Laplace
c)Solve algebraic equation
d) To get back required solution of ODE, find the inverse Laplace transform of algebraic solution.

Study an example:

$$y' + 3y = \sin(\omega t) \tag{2}$$

Let

$$L(y) : \quad Y(p) = \int_0^\infty e^{-pt} y(t) \, dt$$

Multiply Eq. (**??**) by $e^{-pt}\,dt$ and integrate. Hence

$$\int_0^\infty e^{-pt} y'(t)\,dt + 3Y(p) = \frac{\omega}{p^2+\omega^2} \tag{3}$$

where we used our recent example to write down the RHS i.e. $L(\sin)$.

For the first term we use integration by parts:

$$\int_0^\infty e^{-pt} y'(t)\,dt = [e^{-pt} y(t)]_0^\infty + p\int_0^\infty e^{-pt} y(t)\,dt = pY(p) - y(0).$$

Notice that the initial condition of y has made an explicit appearance here.

Hence the algebraic equation resulting from Laplacing the original equation is now:

$$\begin{aligned} (3+p)Y(p) &= y(0) + \frac{\omega}{p^2+\omega^2} \\ Y(p) &= \frac{y(0)}{3+p} + \frac{1}{3+p} \times \frac{\omega}{p^2+\omega^2} \end{aligned} \tag{4}$$

The final step is to take the inverse Laplace transform.

The first term is easy: $y(0)e^{-3t}$. We have worked out

$$\int_0^\infty e^{-pt} e^{-st} dt = 1/(p+s)$$

this in class. In the Laplace Tables we are given

$$L[e^{-at}] = \frac{1}{p+a}, \ldots (L.2)$$

which is exactly our result.

Second term can be done by looking up tables. To do that we need to prepare a bit by using partial fractions.

$$\frac{\omega}{p^2+\omega^2} = \frac{1}{2i}\left(\frac{1}{p-i\omega} - \frac{1}{p+i\omega}\right).$$

So the second term is

$$\frac{1}{2i}\left(\frac{1}{p+3}\frac{1}{p-i\omega} - \frac{1}{p+3}\frac{1}{p+i\omega}\right)\ldots(A)$$

Now from the Laplace tables we find a helpful transform
For $\Re e(p+a) > 0$ and $\Re e(p+b) > 0$, the relevant transform is

$$L[\frac{e^{-at}-e^{-bt}}{b-a}] = \frac{1}{(p+a)(p+b)}, \ldots (L.7)$$

We can immediately read off the function of which Eq. (A) is the Laplace transform. Putting it together with the first term

$$y(t) = y(0)e^{-3t} + \frac{1}{2i}\left(\frac{e^{-i\omega t}-e^{-3t}}{3-i\omega} - \frac{e^{i\omega t}-e^{-3t}}{3+i\omega}\right) \tag{5}$$

That is the final answer.

§**Higher derivatives:**

In the ODE's we encounter $y', y'', y''', \ldots$, so we need to know the systematics of how to take the Laplace Transform. We already solved one case in the last problem:

$$\begin{aligned} L(y') &= \int_0^\infty y'(t)e^{-pt}\,dt \\ &= pY(p) - y(0) \end{aligned} \tag{6}$$

We used integration by parts. Thus the initiial value $y(0)$ is required here.

Caution: Keep clear sight of capital Y versus lowercase y.

Let us take the next case

$$\begin{aligned} L(y'') &= \int_0^\infty y''(t)e^{-pt}\,dt \\ &= pL(y') - y'(0) \end{aligned} \tag{7}$$

We can substitute from Eq. (**??**) and thus

$$L(y'') = p^2Y(p) - py(0) - y'(0). \tag{8}$$

§**A few typical problems- and ideas for solving these:**

Read through Examples 1,2,3,4 pages 440-441. These are good.

§**Problem #9 Page 443:**

With $y(0) = 0$ and $y'(0) = 8$ solve for $y(t)$:

$$y'' + 16y = 8\cos(4t).$$

Solution: Take LT and use Eq. (**??**) to get $L(y'') = p^2 Y(p)$ and hence

$$(p^2 + 16)Y(p) = 8 + \frac{8p}{p^2 + 16}.$$

Hence

$$Y(p) = \frac{8p}{(p^2 + 16)^2} + \frac{8}{p^2 + 16}$$

Now scan the Tables: Use L11 and L3

$$y(t) = (t + 2)\sin(4t).$$

We can check this is right by taking derivatives of the proposed solution:

$$\begin{aligned} f'(t) &= \sin(4t) + 4(2 + t)\cos(4t) \\ f''(t) &= -16(2 + t)\sin(4t) + 8\cos(4t). \end{aligned} \tag{9}$$

Therefore ODE is reproduced. Also the initial values are satisfied.

§**Problem #27 Page 443:**

Warning: We will go over the details carefully, so this discussion will be somewhat longer than strictly necessary!

With $y(0) = 0 = y'(0)$ and with $z(0) = 4/3$, solve

$$\begin{aligned} y' + z' - 3z &= 0 \\ y'' + z' &= 0 \end{aligned} \tag{10}$$

This is interesting since we have coupled equations. Let us say $Y(p)$ and $Z(p)$ are the LT's of y, z respectively. Taking LT's we get a pair of equations

$$\begin{aligned} [pY(p)] + [pZ(p) - 4/3] - 3Z(p) &= 0 \\ [p^2Y(p)] + [pZ(p) - 4/3] &= 0, \end{aligned} \tag{11}$$

where each square bracket is the LT of the individual terms, and we have *already* used the initial conditions.

Rearranging

$$pY(p) + (p-3)Z(p) = \frac{4}{3} \tag{12}$$

$$p^2Y(p) + pZ(p) = \frac{4}{3} \tag{13}$$

The neat thing is we can solve coupled algebraic equations easily. Divide Eq. (**??**) by p and subtract (Eq. (**??**)-Eq. (**??**)). This gives:

$$(p-4)Z(p) = \frac{4}{3} - \frac{4}{3}\frac{1}{p},$$

or

$$Z(p) = \frac{4}{3}\frac{1}{(p-4)} - \frac{4}{3}\frac{1}{(p-4)}\frac{1}{p},$$

Now use

$$\frac{1}{(p-4)}\frac{1}{p} = \frac{1}{4}\left(\frac{1}{(p-4)} - \frac{1}{p}\right)$$

therefore by simplifying

$$Z(p) = \frac{1}{3p} + \frac{1}{p-4}$$

Now we can plug in for $Y(p)$ into Eq. (**??**). In preparation note that taking partial fractions is extremely helpful:

$$(p-3)Z(p) = \frac{p-3}{3p} + \frac{p-3}{p-4} = [\frac{1}{3} - \frac{1}{p}] + \frac{p-4+1}{p-4} = \frac{4}{3} + \frac{1}{p-4} - -\frac{1}{p}.$$

Plugging in we get

$$Y(p) = \frac{1}{p^2} + \frac{1}{4p} - \frac{1}{4(p-4)}.$$

Use L5 for inversion. With $k > -1$

$$L(t^k) = \frac{k!}{p^{k+1}}.$$

Hence

$$\begin{aligned} y(t) &= \frac{1}{4} + t - \frac{1}{4}e^{4t} \\ z(t) &= \frac{1}{3} + e^{4t}. \end{aligned} \tag{14}$$