Physics 116B- Spring 2018

0.5cm **Mathematical Methods 116 B**

0.5cm S. Shastry, June 5, 2018

Introduction to Probability theory

§**Basic Definitions:**

- Random variables: e.g. tossing one die, or two dice, or n dice! (dice is the plural of die!!)
- Sample space. Number of points in sample space N_S
- Events E
- Probability of an event $P(E) = \frac{N_E}{N_S}$.

§**First example Single die:**

Sample space $= h, t$ with $N_S = 2$
Event h or t
Probability of heads $P(h) = 1/2$, tails $P(t) = 1/2$.

§**Next example Two dice:**

Sample space: (just list and then count all possibilities).

$$hh, ht, th, tt$$

$$N_S = 4$$

.

Events: (1) Single h (2) Two h's (3) At least one h etc

$$P(single\, h) = \frac{1}{2}$$

$$P(two\, h's) = \frac{1}{4}$$

$$P(at\,least\,one\,h) = \frac{3}{4}.$$

We thus see that there is an element of "word power" here, precise words are key to many basic problems in this field.

§ **Three dice:**

Sample space:

$$hhh, hht, hth, thh, htt, tht, tth, ttt$$

$$N_S = 8$$

$$\begin{aligned}
P(no\,h) &= 1/8 \\
P(one\,h) &= 3/8 \\
P(two\,h) &= 3/8 \\
P(three\,h) &= 1/8 \\
P(no\ successive, h) &= \frac{1}{4} \\
P(successive\,h) &= \frac{3}{8} \\
P(at\,least\,one\,h) &= \frac{3}{8}
\end{aligned}$$

Hence we can cook up many detailed events that live in the sample space. Every definable event has a calculable probability!

§**Successive events:**

We can have successive events that are either independent or dependent on each other. Both can be considered in a simple context. Let us take 15 billiard balls in a box, 5 are red and 10 are white. We dip into the box and draw out balls of various colors.

§**Independent events problem:**

I want the probability $P(AB)$ of two successive events A with probability $P(A)$, is where I pick a white ball on my first dip. I then replace the ball into the box and then the second event B with probability $P(B)$, is where I pick a white ball on my next dip.

We can prove easily in this case.

$$P(AB) = P(A)P(B) = P(B)P(A)$$

§**Dependent events problem:**

I want the probability $P(AB)$ of two successive events A with probability $P(A)$, is where I pick a white ball on my first dip. I then *do not replace the ball* into the box and then the second event B with probability $P(B)$, is where I pick a white ball on my next dip.

We can check

$$P(AB) = P(A)P_A(B)$$

where $P_A(B)$ is a conditional probability, where event A is supposed to have occurred already.

Similarly

$$P(BA) = P(B)P_B(A) = P(AB).$$

Symmetric...

§**Permutations, Combinations, Number of ways...:**

A classic set of twin problem is as follows:

A club of physics students has 10 members. (More generally $10 \to N$) (Incidentally two of them are John and Sarah)

a)In how many ways can we we choose two office bearers. (More generally $2 \to n$).

b)In how many ways can we we choose two office bearers, one a president and the other the vice president of the club.

Before solving this simple problem, note the difference between the two cases. In (a) we do not worry about what the role of the two members is, while in (b) we do. This leads to different answers.

Solution

(a) Here we want to ignore the rank of the chosen pair. Hence the number of ways is $10 \times 9/2$, i.e. $N_b = \frac{N!}{(N-n)!n!}$. The probability of choosing John and Sarah is therefore 2/90, which is consistent with above.

We call ${}^N C_n = \frac{N!}{(N-n)!n!}$, (usually called N choose n). This is the number of ways of choosing n identical objects out of a group of N.

This is a binomial coefficient since

$$(1+x)^N = \sum_{n=0}^{N} x^n \; {}^N C_n.$$

(b) To choose the first member i.e. the president, we have 10 choices. Having made that choice the second member i.e. the vice-president can be chosen from the rest, i.e. 9 choices.

Hence $N_a = 10 \times 9$. More generally $N_a = \frac{N!}{N-n!}$. This number is called ${}^N P_n \equiv \frac{N!}{(N-n)!}$- check that this is true here.

Connecting to probabilities, this means that the probability of John and Sarah being president and vice president is $\frac{1}{90}$ and the reverse choice also has the same probability.

Another problem: Four boxes are populated (one per box) from a group of red and white billiard balls that are mixed up thoroughly, so each time we pick a ball, it could be equally red or white.

The sample set of this consists of 16 states

$$rrrr, rrrw, rrwr, rwrr, wrrr, rrww, rwrw, rwwr$$

$$and\ 8\ more\ by\ switching$$

$$r \leftrightarrow w$$

What is the probability of finding n reds, where $n = 0, 1, \ldots, 4$?

Answer: $P(r) =^4 C_r \, \frac{1}{16}$. (Recall $\sum_{r=0,4} {}^4C_r \ = 16$ is the total number of configurations.)

$$r = 0, \ P = 1/16$$

$$r = 1, \ {}^4C_1 = 4, \ P = 1/4$$

$$r = 2, \ \ {}^4C_2 = 6, \ \ P = 3/8$$

$$r = 3, \ \ {}^4C_3 = 4, \ \ P = 1/4$$

$$r = 4, \ \ {}^4C_4 = 6, \ \ P = 1/16$$

Total P=1.

§ **The urn problem of Maxwell-Boltzmann, Bose-Einstein, and Fermi-Dirac:**

§ **Sample space for putting 2 particles in 3 boxes:**

More generally $3 \to N$ and $2 \to n$.

Comment: In quantum statistical mechanics, the boxes are the states into which we put our particles.

a) Maxwell-Boltzmann

Labelled particles- say "r" and "b" and no restrictions:

Sample space:

$$rb|0|0,\;\; 0|rb|0,\;\; 0|0|rb,\;\; r|b|0,\;\; b|r|0,\;\; 0|r|b,\;\; 0|b|r,\;\; r|0|b,\;\; b|0|r$$

9 configurations.

Note that although $r|b|0$ and $b|r|0$ are counted separately, we do not distinguish between $rb|0|0$ and $br|0|0$.

(b) Bose-Einstein:

Particles have lost all labels, i.e. identical particles. No other restrictions.

Sample space:

$$xx|0|0,\; x|x|0,\; x|0|x,\; 0|xx|0,\; 0|x|x,\; 0|0|xx$$

6 configurations.

$$\nu = \frac{(N-1+r)!}{r!(N-1)!}$$

($S = k_B \log \nu$ is the entropy of this system).

(c) Fermi-Dirac:

Particles have lost all labels, i.e. identical particles. Single occupancy restriction.

$$x|x|0,\; x|0|x,\; 0|x|x,$$

3 configurations.

$$\nu = \frac{N!}{n!(N-n)!}$$

($S = k_B \log \nu$ is the entropy of this system too.)

§Random variables and probability distribution: Discrete case:

Example: $x = \pm 1$ with $P(1) = .6$ and $P(-1) = .4$. This is a loaded coin. $P = \frac{1}{2}$ for both would be a unloaded coin.
Similarly we can think of a m-state die where

$$x = \{x_1, x_2, \dots x_m\}$$

with probabilities $P(x_j) =$

$$P(x_j) = \{p_1, p_2, \dots p_m\}$$

.

We must have two conditions for this to be a well defined probability:

$$P(x_j) \geq 0, \ \ \text{and} \ \sum_j P(x_j) = 1$$

We can define the mean, the variance and the standard deviation as follows.

$$\sum_j x_j P(x_j) = \bar{x} = \text{Mean value of x}$$

$$\sum_j x_j^2 P(x_j) = \bar{x^2} = \text{Mean square value of x}$$

We can define the variance

$$\text{Variance} = Var(x) = \bar{x^2} - (\bar{x})^2$$

We can define the Standard deviation as

$$\text{Standard Deviation} = \sqrt{Var(x)} = \sqrt{\bar{x^2} - (\bar{x})^2}$$

§Random variables and probability distribution: Continuous case: The Gaussian:

$$P_G(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$$

$$\bar{x^0} = \int_{-\infty}^{\infty} dx\, P_G(x) = 1.$$

$$\bar{x} = \int_{-\infty}^{\infty} dx\, x P_G(x) = \mu$$

$$\bar{x^2} = \int_{-\infty}^{\infty} dx\, x^2 P_G(x) = \mu^2 + \sigma^2$$

Hence Variance of the Gaussian is σ^2 and the standard deviation is σ.

More convenient notation is

$$\langle A \rangle = \bar{A}.$$

Note the general result:

$$\langle (x - \langle x \rangle)^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2$$

Key result: Central Limit theorem.

We consider a composite variable

$$y = \sum_{j=1}^{N} x_j,$$

where $N \gg 1$ and the x_j have any distribution whatsoever, and all we ask is that these are un-correlated.

Question: What does that mean?

In such a case, the central limit theorem says

$$\lim_{N \gg 1} P(y) \to P_G(y)$$

where $P_G(y|\mu, \sigma)$ depends on some μ, σ, which can be found most easily from experiments. The point is that y acts as a Gaussian variable.

In experimental physics, and also in engineering, this is a crucial result.