Physics 116B- Spring 2018

Mathematical Methods 116 B

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Notes on Orthogonal Curves

§Orthogonal Family of Curves:

This is a part of Section 2, page 396 in the Boas book. The main idea here is to understand the meaning of the constant of integration and the solution of the ODE in a geometric fashion.

Let us say we solve the ODE

$$\frac{dy}{dx} = -\frac{P(x,y)}{Q(x,y)} \tag{1}$$

with some given P, Q. This gives us a solution which we can write as

$$y(x) = y_1(x, A),$$

where $y_1(x, A)$ is some specific function of x and a constant A of integration. This function depends on P, Q and let us imagine we have an explicit solution.

For a fixed A we can now plot y versus x and get some curve in the $x - y$ plane. Now by taking different values of A we get another curve, etc. Let us take the collection of these curves. They are called a *family of curves.*

Now we write down another differential equation, closely related to Eq. (1)

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)}. \tag{2}$$

We could solve this as well and write the solutions in the form

$$y(x) = y_2(x, B),$$

where $y_2(x, B)$ is another specific function of x, and a constant B of integration. We can proceed as above and take the collection of $y - x$ curves with different choices of B and call them the second family of curves.

Clearly at a point (x, y) in the 2-d plane, which is common to both equations (i.e. an intersection point), we have the property

$$(\frac{dy}{dx})_{\text{Eq. 1}}(\frac{dy}{dx})_{\text{Eq. 2}} = -1. \tag{3}$$

(Recall orthogonal lines $m_1 m_2 = -1$)

This means that if *any* solution of Eq (1) (with a specific choice of A), intersects with a solutions of Eq (2) (with a particular choice of B), they will intersect at right angles. Hence the two families represent orthogonal sets of curves.

A Differential Equation is given to us as
y^2dx+ x dy=0
This corresponds to some family of curves, where the slope is given by

dy/dx= - y^2/x

Integrating, we find

x = X0 $e^{1/y}$
where X0 is the constant of integration. By varying X0 we get different curves. This is called a family of curves. (See below Plot name is fns)

We want to find the orthogonal family of curves to this family.

We find the orghogonal family by setting its slope to be inverse negative of the slope of the above family

Hence dy/dx= x/y^2

This family can also be solved and we find

$y^3/3 - x^2/2 = -\text{Y0}/2$
where Y0 is another constant of integration. Varying Y0 gives rise to *a* distinct family of curves.

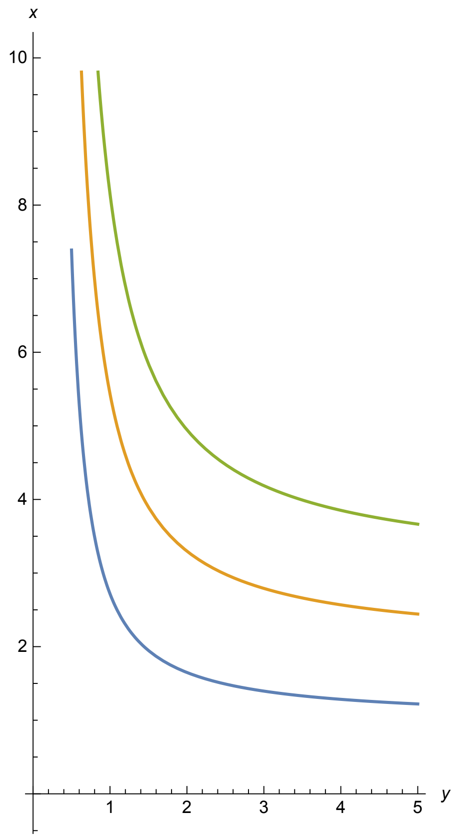
Let us check what these curves look like. By above argument they should be orthogonal to the other family found above – let us see if this is so.

In the curves below we write X0 -> X and Y0 -> Y.

```
firstset[X_, y_] = X Exp[1 / y]
```

$e^{\frac{1}{y}}$ X

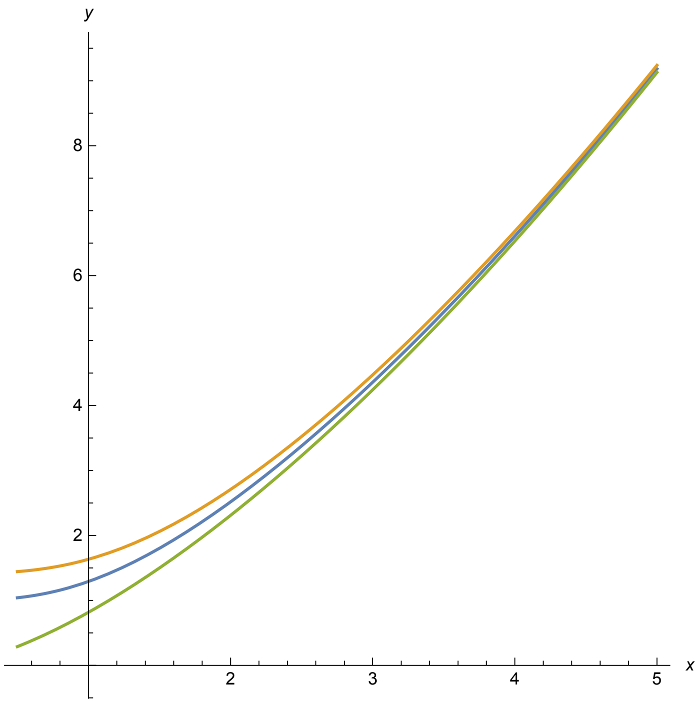
```
fns = Plot[ {firstset[1, y], firstset[2, y], firstset[3, y]},
  {y, .5, 5}, AxesOrigin → {0, 0}, AspectRatio → 2, AxesLabel → {y, x}]
```




```
secondset[Y_, y_] = Sqrt[2 y^3 / 3 + Y]
```

$$\sqrt{\frac{2\,y^3}{3} + Y}$$

```
perp = Plot[ {secondset[1, y], secondset[2, y], secondset[0, y]},
  {y, .5, 5}, AspectRatio → 1, AxesLabel → {x, y}]
```




```
Show[fns, perp]
```