Physics 116B- Spring 2018

Mathematical Methods 116 B

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Series solution of Differential Equations
Legendre, Bessel equations

§**Differential equations arising often in applications:** Differential

equations of great importance have the form

$$h(x)y'' + f(x)y' + g(x)y = 0,$$

- *Legendre Equation* $h(x) = 1 - x^2$, $f(x) = -2x$, $g(x) = l(l+1)$. Integer l
- *Bessels equations* $h(x) = x^2$, $f(x) = x$ and $g(x) = x^2 - p^2$. Integer p

§**Basic scheme of series solution:** i) Assume a power series

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots,$$

ii) Find recursion relations for a_n.
iii) Solve recursion relation by iteration.

§**Warm-up with a simpler equation:**

$$y' + 2xy = 0,$$

Plug in $y = \sum_{n=0}^{\infty} a_n x^n$. Hence $y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$, and hence

$$y' + 2xy = \sum_{n=0}^{\infty} n a_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^{n+1}$$

$$= \left(a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \ldots\right) + 2\left(a_0 x + a_1 x^2 + a_2 x^3 + a_3 x^4 + \ldots\right)$$

$$0 = (a_1) + x(2a_2 + 2a_0) + x^2(3a_3 + 2a_1) + x^3(4a_4 + 2a_2) + \ldots.$$

Now we equate each power of x to zero. §**Why ?**

$$a_1 = 0, \;\; 2(a_2 + a_0) = 0, \;\; (3a_3 + a_1) = 0, \;\; (4a_4 + 2a_2) = 0, \ldots$$

Odd indices

$$a_{2m+1} = 0,$$

because $a_1 = 0$ (only one term with power x^1).!!
Even indices. Call $\alpha_n = \frac{a_n}{a_0}$
Recursion relation

$$a_{m+2} = -\frac{2}{m+2} a_m, \;\; m = 0, 2, 4, \ldots$$

$$\alpha_2 = -2/2, \;\; \alpha_4 = 4/(2.4), \;\; \alpha_{2n} = (-1)^n/n!$$

Hence the solution is

$$y = a_0 e^{-x^2}.$$

§**Legendre equation:**

$$(1 - x^2)y'' - 2xy' + l(l + 1)y = 0,$$

Plug in

$$y = a_0 + a_2x^2 + a_4x^4 + \ldots + a_1x + a_3x^3 + a_5x^5 + \ldots$$

Calculate and find recursion relations

$$a_{n+2} = -a_n \times \frac{(l - n)(l + n + 1)}{(n + 2)(n + 1)},$$

With two starter values a_0, a_1 we get the rest in terms of these.

$$y = a_0 \left(1 - \frac{l(l + 1)}{2!}x^2 + \frac{l(l + 1)(l - 2)(l + 3)}{4!}x^4 + \ldots\right)$$
$$+a_1 \left(x - \frac{(l - 1)(l + 2)}{3!}x^3 + \ldots\right)$$

Thus we get two series with two starter coefficients, which we can tune as we like. The series converge for $|x| < 1$ and diverge for $|x| \geq 1$ from the ratio test.

§**Truncation of series:**

If l is an integer, one of the two series truncates. We get polynomials

- $l = 0$ We set $a_1 = 0$ so that
$$y = a_0$$
- $l = 1$ We set $a_0 = 0$ so that
$$y = a_1x$$
- $l = 2$ We set $a_1 = 0$ so that
$$y = a_0(1 - 3x^2)$$
- $l = 3$ We set $a_0 = 0$ so that
$$y = a_1(x - 5/3x^3)$$

If we normalize these polynomials to $P_n\big|_{x=1} = 1$, we get the Legendre polynomials.

$$P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{1}{2}(3x^2 - 1), \ldots$$

In general $P_l(x)$ is a polynomial in x of degree l.

§**These satisfy Rodrigue's formula:**

$$P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2 - 1)^l$$

§ **Generating Function:** These can be found from a neat generating formula valid for $|h| < 1$

$$\Phi(x, h) = (1 - 2xh + h^2)^{-\frac{1}{2}}.$$

$$\Phi(x, h) = \sum_{l=0}^{\infty} h^l P_l(x).$$

Origin of Legendre polynomials and the generating function. Connection to multipole expansion.

Consider two charges located at $\vec{r}$ and $\vec{r'}$ and let $r = |\vec{r}| \gg r'(= |\vec{r'}|)$. Coulomb potential

$$U(\vec{r}, \vec{r'}) = \frac{e^2}{|\vec{r} - \vec{r'}|} = \frac{e^2}{\sqrt{r^2 + r'^2 - 2\vec{r}.\vec{r'}}} \tag{1}$$

If we call $x = \vec{r}.\vec{r'}/(rr') = \cos(\theta)$, then $-1 \le x \le 1$. Let us call

$$h = r'/r,$$

and assuming $h \ll 1$ we now expand to generate the multipole expansion.

$$\begin{aligned} U(\vec{r}, \vec{r'}) &= \frac{e^2}{r}\frac{1}{\sqrt{1 + h^2 - 2hx}} = \frac{e^2}{r}\Phi(x, h) \\ &= \frac{e^2}{r}\sum_{l=0}^{\infty} h^l P_l(x). \\ &= e^2\left(\frac{1}{r} + \frac{r'}{r^2}P_1(x) + \frac{r'^2}{r^3}P_2(x) + \ldots\right). \end{aligned} \tag{2}$$

Reminder

$$P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{1}{2}(3x^2 - 1), \ldots$$

§**Recursion relations:**

$$xP_n'(x) = P_{n-1}'(x) + nP_n(x) \quad (R - I)$$

$$nP_n = (2n-1)xP_{n-1} - (n-1)P_{n-2} \quad (R - II)$$

e.g. we can use the second of these to calculate P_2 from P_0, P_1

$$2P_2 = 3xP_1 - P_0 = 3x^2 - 1. \text{ Works}$$

§**Usage of generating function to find Recursion relations:**

$$\Phi(x,h) = (1 - 2xh + h^2)^{-\frac{1}{2}}.$$

$$\Phi(x,h) = \sum_{l=0}^{\infty} h^l P_l(x).$$

(I) Take the derivative

$$\frac{\partial \Phi(x,h)}{\partial h} = \frac{x-h}{1-2hx+h^2}\,\Phi(x,h)$$

Now cross-multiply and plug in the expansion and equate powers of h^n

$$(n+1)P_{n+1} - 2xnP_n + (n-1)P_{n-1} = xP_n - P_{n-1},$$

i.e.

$$(n+1)P_{n+1} - (2n+1)xP_n + nP_{n-1} = 0.$$

This is $(R - II)$.

(II) Take the derivative

$$\frac{\partial \Phi(x,h)}{\partial x} = \frac{h}{1-2hx+h^2}\,\Phi(x,h)$$

Now cross-multiply and plug in the expansion and equate powers of h^n

$$P_n' - 2xP_{n-1}' + P_{n-2}' = P_{n-1}$$

§**Orthonormality:**

$$\int_{-1}^{1} P_l(x)P_m(x)\,dx = \delta_{l,m}\frac{2}{2l+1}.$$

We can show this using the differential equation.

$$\frac{d}{dx}[(1-x^2)P_l'(x)] + l(l+1)P_l(x) = 0$$

$$\{l(l+1) - m(m+1)\}P_lP_m = P_l\frac{d}{dx}[(1-x^2)P_m'] - P_m\frac{d}{dx}[(1-x^2)P_l']$$

$$= \frac{d}{dx}[(1-x^2)(P_lP_m' - P_l'P_m)]$$

Integrate w.r.t. x assuming $l \neq m$.

$$\int_{-1}^{1} dx\frac{d}{dx}[(1-x^2)(P_lP_m' - P_l'P_m)] = [(1-x^2)(P_lP_m' - P_l'P_m)]_{-1}^{1} = 0.$$

Hence

$$\int_{-1}^{1} dxP_lP_m = 0, \quad \text{if} \quad l \neq m.$$

For $l = m$ further simple calculation shows the important orthonormality result.

§**Use of the orthonormality of the Legendre polynomials. The Legendre series:**

A highly useful result follows. Any function $f(x)$ on the interval $-1 \leq x \leq 1$ can be expanded

$$f(x) = \sum_{j=0}^{\infty} c_j P_j(x),$$

where

$$c_j = \frac{2j+1}{2} \int_{-1}^{1} dx f(x) P_j(x)$$

If $f(x)$ is a polynomial of degree r, the c_j vanish for $j > r$.
Example:
Problem (A)

$$F(x) = 2x^2 + x - 1$$

Answer

$$c[0] = -1/3, c[1] = 1, c[2] = 4/3, c[3] = 0, \ldots$$

Problem (B)

$$F(x) = |x| + 3x^4$$

Even function hence $c[2n+1] = 0$
Answer:

$$c[0] = 11/10, c[2] = 131/56, c[4] = 279/560, c[6] = 13/128, c[8] = -17/256$$

Problem (C)

$$F(x) = x^2 + 3x^4$$

Even function hence $c[2n+1] = 0$
Answer:

$$c[0] = 14/15, c[2] = 50/21, c[4] = 24/25, c[6] = 0, c[8] = 0$$

§**Bessel Functions- the bare facts:**

$$x^2y'' + xy' + (x^2 - m^2)y = 0$$

we solve it with a series

$$y = x^m \left(1 - \frac{1}{m+1}(\frac{x^2}{2^2}) + O(x^4)\dots\right)$$

This is essentially a Bessel function....

$$J_m(x) = \frac{1}{m!}(\frac{x}{2})^m \left(1 - \frac{1}{m+1}(\frac{x^2}{2^2}) + O(x^4)\dots\right)$$

Why the starting index x^m?

Plug in $y \sim x^r$ and differentiate and cancel x^r. At small r we get a regular solution

$$r^2 = m^2,\ r = \pm m$$