Physics 116B- Spring 2018

Mathematical Methods 116 A

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Notes on Tensors

§ **Scalars, Vectors, Tensors,..., Pseudo Scalars, Pseudo Vectors, Pseudo Tensors:**

These are defined by the behavior under rotations and reflections of various physical quantities.

§**Refresher on Notation and Rotations:**

Let us take 3-d and write a point in space represented by $\vec{r}$, a vector joining it to the origin O. We will write the components of this vector in the form

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

where $\hat{i}$ etc are unit vectors in the three directions. For convenience we will map

$$\begin{aligned} \hat{i} &\to \hat{e}_1, \quad x \to x_1 \\ \hat{j} &\to \hat{e}_2, \quad y \to x_2 \\ \hat{k} &\to \hat{e}_3, \quad z \to x_3 \end{aligned}$$

and so our vector can be written compactly as

$$\vec{r} = \sum_j \hat{e}_j x_j \to \hat{e}_j x_j, \tag{1}$$

where we introduce the Einstein convention to simplify writing.

In any equation with (tensor) indices, sum over any repeated index, the sum being over its natural range.

We will take the "passive viewpoint", i.e. keep the point we are describing fixed, and rotate the frame of reference. The new frame gives us new basis vectors $\hat{e}_j$ and new coordinates x'_j. This means that in the new frame of reference the *same point* can be written as

$$\vec{r} = \hat{e}'_j x'_j \tag{2}$$

Thus comparing the two equations

$$\vec{r} = \hat{e}'_j x'_j = \hat{e}_j x_j. \tag{3}$$

Note that the new frame is reached by some rotation or reflection and hence it remains orthogonal, or to be more precise, a *cartesian frame* defined by the condition

$$\hat{e}_i.\hat{e}_j = \delta_{ij} = \hat{e}'_i.\hat{e}'_j.$$

Here an later we use the Kronecker delta function

$$\begin{aligned} \delta_{ij} &= 1, \text{ if } i = j \\ &= 0, \text{ if } i \neq j. \end{aligned} \tag{4}$$

In order to fill in the details, I would need to express the new basis vectors in terms of the old ones. Thus we need the transformation rules

$$\begin{aligned} \hat{e}'_1 &= l_1\hat{e}_1 + m_1\hat{e}_2 + n_1\hat{e}_3 \\ \hat{e}'_2 &= l_2\hat{e}_1 + m_2\hat{e}_2 + n_2\hat{e}_3 \\ \hat{e}'_3 &= l_3\hat{e}_1 + m_3\hat{e}_2 + n_3\hat{e}_3 \end{aligned} \tag{5}$$

Question: What is the meaning of the coefficients? (Figure)

A matrix method of describing the same transformation next. Let us rewrite the transformation of the components as

$$\begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} = A. \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

where

$$A = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}$$

We have seen that the rotation matrices A can be easily calculated for rotations about the main axes, e.g. a rotation about the z axis by angle θ gives us

$$A_z(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Similarly we can write the rotation about the old y axis

$$A_y(\gamma) = \begin{bmatrix} \cos(\gamma) & 0 & \sin(\gamma) \\ 0 & 1 & 0 \\ -\sin(\gamma) & 0 & \cos(\gamma) \end{bmatrix}.$$

In fact an important theorem, which we will not pursue here, is due to Euler who showed that *an arbitrary rotation* can be written in the form

$$R(\alpha, \beta, \gamma) = A''_z(\gamma) A'_y(\beta) A_z(\alpha)$$

where A'_y is a rotation about the *new* y axis (after performing the first z rotation) and A''_z is a rotation about the *newest* y axis (after performing the second y rotation). The three angles α, β, γ are known as the Euler angles.

We will bypass the details of this construction- it is the core part of a graduate course- and merely note that for all rotations

$$Det(A) = 1.$$

If we also add in a reflection, the transformation leads to matrices such as

$$\Pi = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Clearly the determinant is now -1.

Let us note that the rotation matrices are orthogonal,

$$A^T A = \mathbb{1}.$$

This remains true if we add reflections.

§**Reconnecting and simplifying the notation:**

In the tensor notation we map the transformation matrix

$$A \to a_{ij}$$

and

$$A^T \to a_{ji}$$

We can thus write for any vector $\vec{V} = V_j \hat{e}_j$ the transformation $V \leftrightarrow V'$ under A or its inverse is simply written

$$V_i' = a_{ij} V_j$$

and the inverse is

$$V_i = a_{ji} V_j'.$$

Imagine next we had an object that is the direct product of two vectors

$$Q_{ij} = V_i V_j,$$

and we will see many examples soon in the physical world. Then it will transform as

$$Q_{ij}' = a_{ik} a_{jl} Q_{kl}.$$

This is the transformation property of a second rank tensor!!

§**Definitions:** We will define various objects under proper rotations ($DetA = 1$). Improper rotations ($DetA = -1$) involve reflections plus rotations.

- Scalar - unchanged under proper rotations.

  * Energy, mass, total charge

- Vector- Transforms as a vector under proper rotations. $V_i' = a_{ij} V_j$

  * Position vector, momentum, angular momentum (pseudo vector), dipole moment of a charge distribution

- Second Rank Tensor Transforms as a direct product of two vector under proper rotations. $T_{ij}' = a_{ik} a_{jl} T_{kl}$

  *Strain $\frac{\partial u_i}{\partial x_j}$, where u_i is the displacement. Also stress, moment of inertia, quadrupole moment of a charge distribution

- Third Rank Tensor Transforms as a direct product of three vector under proper rotations. $T'_{ijm} = a_{ik}a_{jl}a_{mn}T_{kln}$

  * Higher multipole moment of a charge distribution.

§**Examples of notation and contraction:**

- $a_{ii} = ?$
- $a_{ij}b_{jk} = ?$
- $\frac{\partial u}{\partial x_j}\frac{\partial x_j}{\partial x_k} = ?$
- Let us define a transformed 4th rank tensor $T'_{ijkl} = a_{i\alpha}a_{j\beta}a_{k\gamma}a_{l\delta}T_{\alpha\beta\gamma\delta}$. What is the meaning of $T'_{ijkj} = ?$ Reduces rank to 2 from 4.

§**Comments:**

- Tensors- matrices. Rank1-vectors, Rank2-matrices
- Symmetric and antisymmetric tensors: $T_{ij} = \pm T_{ji}$. Can decompose any tensor into symmetric + antisymmetric pieces.
- Can add tensors of the same rank, not otherwise. $T_{ij} + A_l A_m Q_{limj} = ?$

  * BTW, is $A_l A_m Q_{limj} =^? A_l A_m Q_{lmij}$

§ **Moment of Inertia tensor:**

This is a good example of a second rank tensor. To remind you, when a body rotates, it has angular momentum and the Newtonian equation says $\dot{\vec{L}} = \vec{N}$ the torque. At zero torque, we must have a constant (i.e. time independent) $\vec{L}$. We equate this to its angular velocity $\vec{\omega}$, the proportionalty constant is the moment of inertia

$$L_i = I_{ij}\omega_j.$$

Clearly I_{ij} must be a tensor of rank 2 to balance the equation.

For a point mass rotating about the origin, we can be more explicit. Let the position of the mass be $\vec{r}$, whose magnitude is fixed (rigid rotation) but the angular position varies. Thus

$$\vec{v} = \dot{\vec{r}}, \text{ with } \vec{v}.\vec{r} = 0, \text{ so that } d(\vec{r}.\vec{r})/dt = 0.$$

We then define the angular velocity via

$$\vec{\omega} \times \vec{r} = \vec{v},$$

so that its dot product with $\vec{r}$ vanishes. Hence

$$\vec{L} = m\vec{r} \times \vec{v} = m\vec{r} \times (\vec{\omega} \times \vec{r}) = m\left[\vec{\omega}(\vec{r}.\vec{r}) - \vec{r}(\vec{\omega}.\vec{r})\right].$$

We see later that the tensor indices help us evaluate such triple products more easily. For now let us write this in component form

$$\{L_x, L_y, L_z\} = m\{\omega_x(y^2+z^2)-\omega_y xy-\omega_z xz, \omega_y(x^2+z^2)-\omega_x yx-\omega_z xz, \omega_x(x^2+y^2)-\omega_z zx-\omega_y yx\}$$

We can read off the components of the moment of inertia tensor, which is shown to be symmetric (hence less writing!!)

$$I_{zz} = m(x^2 + y^2), \quad I_{zy} = -mzy, \quad I_{zx} = -mzx$$

For a set of particles we sum over each particle and hence

$$I_{zz} = \sum_i m_i(x_i^2 + y_i^2), \quad I_{zy} = -\sum_i m_i z_i y_i, \quad I_{zx} = -\sum_i m_i z_i x_i$$

If the body is viewed as having a density of masses

$$M = \int d^3r \rho(x),$$

then we can generalize and write

$$I_{zz} = \int d^3r\ m(r)(x^2+y^2), \quad I_{zy} = -\int d^3r\ m(r)\ zy, \quad I_{zx} = -\int d^3r\ m(r)\ zx.$$

§**A few problems involving the MOI:**

§**Levi-Civita symbol:**

{ Tullio Levi-Civita }

We define a useful tensor in 3 dimensions

$$\epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{ikj}$$

It is more completely defined by saying any permutation $ijk = P(123)$ gives $|\epsilon_{ijk}| = 1$ otherwise it is zero. The sign is ± 1 depending on the signature of the permutation

§It is useful for defining cross products and all kinds of vector identities.

$$\vec{A} \times \vec{B} = \vec{C}$$

implies

$$C_i = \epsilon_{ijk} A_j B_k.$$

We note that

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta kn - \delta_{jn}\delta_{km}$$

§Let us use Levi-Civita to express the triple cross:

$$\begin{aligned} [\vec{A} \times (\vec{B} \times \vec{C})]_i &= \epsilon_{ijk} A_j (\vec{B} \times \vec{C})_k \\ &= \epsilon_{ijk}\epsilon_{klm} A_j B_l C_m \end{aligned}$$

But $\epsilon_{ijk} = \epsilon_{kij}$ by cyclic permutation, and hence we can write

$$(LHS)_i = \epsilon_{kij}\epsilon_{klm} A_j B_l C_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) A_j B_l C_m = B_i (A_j C_j) - C_i (A_j B_j).$$

Therefore it follows that

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A}.\vec{C}) - \vec{C}(\vec{A}.\vec{B}) \tag{6}$$

§It is also useful for defining the determinant in 3 dimensions.

$$Det A \; \epsilon_{ijk} = A_{il} A_{jm} A_{kn} \epsilon_{lmn}.$$