Physics 116B- Spring 2018

Mathematical Methods 116 A

S. Shastry, April 10, 2018
Notes on Tensors-II

§ Quotient Rule for Tensors: Rapid summary :

Two vectors U, V are related by

$$U_i = T_{ij} V_j. \quad \{1.\}$$

If *every* vector V is mapped into a vector U, then T_{ij} is a second rank tensor.

Here we recall that a vector V transforms under rotation as $V \to V'_i = a_{ij} V_j$ and likewise with U, where a is a rotation matrix.

The proof is straightforward, please read the book. The main thing to keep in mind is that we are trying to show that under the rotation a, T must transform as

$$T'_{ij} = a_{ik} a_{jl} T_{kl},$$

and for this purpose the validity of Eq. (1) for *every* V is necessary.

§ Brief introduction to partial derivatives.:

Since we skipped Chapter 4 in Boas's book, we need to master some elementary facts about partial derivatives.

Recall the (standard) derivative of a function $f(x)$ of a single variable

$$f'(x) \equiv \frac{df(x)}{dx} = \lim_{\epsilon \to 0} \frac{f(x+\epsilon) - f(x)}{\epsilon}$$

Examples:
Now if we have a function of two variables

$$f(x, y),$$

we can define a partial derivative

$$f_x(x,y) \equiv \frac{\partial f(x,y)}{\partial x} = \lim_{\epsilon\to 0}\frac{f(x+\epsilon,y)-f(x,y)}{\epsilon}$$

and likewise with the y derivative, and also higher order derivatives:

$$f_y(x,y) \equiv \frac{\partial f(x,y)}{\partial y} = \lim_{\epsilon\to 0}\frac{f(x,y+\epsilon)-f(x,y)}{\epsilon}.$$

Examples:
1)Separable case: $f = e^x \cos(y)$

$f_x = e^x \cos(y)$; $f_y = -e^x \sin(y)$; $f_{yy} = -e^x \cos(y)$; $f_{xy} = f_{yx} = -e^x \sin(y)$; $f_{xx} =?$

2)Non-separable case: $f = e^{x-y^2}\cos(xy)$

$$f_x = e^{x-y^2}\{\cos(xy) - y\sin(xy)\}.$$

$$f_y = e^{x-y^2}\{-2y\cos(xy) - x\sin(xy).\}$$

Rule:
While taking partial derivative w.r.t. a particular variable, think of **every other** *variable as a constant, and take the usual derivative.*

§**Some important partial derivatives in tensor analysis:** From definition

$$x'_i = a_{ij}x_j, \quad x_i = a_{ji}x'_i,$$

we will calculate

$$\frac{\partial x'_i}{\partial x_j} =?$$

and

$$\frac{\partial x_i}{\partial x'_j} =?$$

§ **Vector calculus made easier using tensor notation:**

We come across the ∇ operator often in E&M, elasticity theory etc

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

Let us rewrite this as

$$\vec{\nabla} = \hat{e}_i \frac{\partial}{\partial x_i},$$

with the usual change of notation $\hat{e}_i = \hat{i}$ etc.

Therefore we can write a few vector calculus relations in compact tensor notation

$$\vec{\nabla}.\vec{\nabla} = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i}$$

$$\vec{\nabla} \times \vec{A} = \hat{e}_i \times \hat{e}_j \frac{\partial A_j}{\partial x_i}.$$

Therefore on taking the k^{th} component

$$\left(\vec{\nabla} \times \vec{A}\right)_k = \epsilon_{ijk} \frac{\partial A_j}{\partial x_i}.$$

The book gives a more easy-to-remember, but completely equivalent relabeling of this equation:

$$\left(\vec{\nabla} \times \vec{A}\right)_i = \epsilon_{ijk} \frac{\partial}{\partial x_j} A_k,$$

where the A_k is written to the right of the x_j derivative.

A nice application is given in the book in (5.14)

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla}.\vec{A}) - (\vec{\nabla}.\vec{\nabla})\vec{A}.$$

§**Dual tensor:**

In 3-dimensions we can write any anti-symmetric tensor in terms of a single vector!

Antisymmetric tensor

$$T_{ij} = -T_{ji}$$

has 3 non-zero components $T_{12}, T_{1,3}, T_{23}$, the diagonals are zero and the other components are found from $T_{21} = -T_{12}$ etc. We can call $T_{12}, T_{1,3}, T_{23} \rightarrow V_1, V_2, V_3$, or more elegantly as

$$T_{ij} = \epsilon_{ijk} V_k.$$

The vector V is called the *dual* of the tensor T. Note this happens only in 3-dimensions.

Moment of inertia revisited:

$$\vec{L} \leftrightarrow \vec{\omega}$$

Constitutives:

$$\vec{L} = m\vec{r} \times \vec{v}, \quad L_i = m\epsilon_{ijk} x_j v_k$$

$$\vec{v} = \vec{\omega} \times \vec{r}, \quad x_i = \epsilon_{ijk} \omega_j x_k$$

Hence

$$L_i/m = \epsilon_{ijk} \epsilon_{klm} x_j \omega_l x_m$$

or

$$L_i/m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) x_j \omega_l x_m$$

Hence

$$I_{il} = m(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) x_j x_m$$

We may rewrite this component-wise i.e. *without* the Einstein summation convention:

$$I_{ii} = m \sum_{j \neq i} x_j^2,$$

and for $i \neq j$

$$I_{ij} = -m x_i x_j.$$

§**Pseudo tensors and pseudo vectors:**

We have discussed the difference between physical quantities under rotations versus rotations plus reflections. On this basis we can distinguish between regular and pseudo objects. Notation in physics often replaces pseudo by axial (pseudo) and polar (regular). These are not computational topics, but rather those of classifications. For example parity was a holy cow until it was shown that parity is violated in weak interactions in 1956-57. C N Yang and T D Lee won the Nobel prize for a theory that predicted parity violation. Ms. Chien-Shiung Wu performed experiments in the same year to confirm the predictions.

Vectors:

$$\vec{r}, \vec{p}$$

Pseudo vectors

$$\begin{aligned} \vec{\omega}: \quad \vec{v} &= \vec{\omega} \times \vec{r}, \\ \vec{L} &= \vec{r} \times \vec{p}. \end{aligned}$$

Pseudo-tensor

$$\epsilon_{ijk}, \quad \text{recall} \;\; L_i = \epsilon_{ijk} r_j p_k$$

§**Curvilinear orthogonal co-ordinates:**

$$\vec{s} = \text{displacement from origin}$$

The arc is defined as

$$d\vec{s} = \hat{i}dx + \hat{j}dy + \hat{k}dz \quad \{Eq2\}$$

Change from $x-y-z$ to curvilinear coordinates. Examples are cylindrical and spherical co-oridnates, and "worse", i.e. more complicated cases!!

Let us focus on one case: cylindrical coordinates.

We change to r, θ, z given as

$$x = r\cos(\theta), \quad y = r\ \sin(\theta), \quad z$$

Hence

$$dx = \cos(\theta)\, dr - r\sin(\theta)\, d\theta$$

$$dy = \sin(\theta)\, dr + r\cos(\theta)\, d\theta.$$

Hence we can rewrite

$$d\vec{s} = \hat{e}_r\, dr + \hat{e}_\theta\, r d\theta + \hat{e}_z dz \quad \{Eq3\}$$

where

$$\begin{aligned} \hat{e}_r &= \hat{\imath}\cos\theta + \hat{\jmath}\sin\theta \\ \hat{e}_\theta &= -\hat{\imath}\sin\theta + \hat{\jmath}\cos\theta \\ \hat{e}_z &= \hat{k} \end{aligned}$$

It is easily seen that these vectors are unit vectors

$$\hat{e}_r.\hat{e}_r = 1, \;\; \hat{e}_\theta.\hat{e}_\theta = 1, \;\; \hat{e}_z.\hat{e}_z = 1.$$

They are also orthogonal to each other:

$$\hat{e}_r.\hat{e}_\theta = 0, \;\; \hat{e}_\theta.\hat{e}_z = 0, \;\; \hat{e}_r.\hat{e}_z = 1.$$

Hence this is another orthonormal set of vectors. But these are curvilinear, and not rectangular.

Pictures:

Now calculate the length of the arc:

$$d\vec{s}.d\vec{s} \equiv ds^2 = dx^2 + dy^2 + dz^2,$$

in the new coordinates from { Eq.3 }

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2.$$

Here r plays the role of a *scale factor* in the second term. Let us picture this equation.

More generally if we change variables from x, y, z to another triad x_1, x_2, x_3 (note that $x_1 \neq x$ now, but rather x_j are some generalized co-ordinates.

We can then write

$$dx = \frac{\partial x}{\partial x_j} dx_j$$

and similarly for dy, dz. Hence from { Eq 2.}

$$\begin{aligned} d\vec{s} &= \hat{i}dx + \hat{j}dy + \hat{k}dz \\ &= \vec{a}_1 dx_1 + \vec{a}_2 dx_2 + \vec{a}_3 dx_3 \ \ \{Eq4.\} \end{aligned}$$

Here

$$\vec{a}_n = \hat{i}\frac{\partial x}{\partial x_n} + \hat{j}\frac{\partial y}{\partial x_n} + \hat{k}\frac{\partial z}{\partial x_n}$$

Now

$$\vec{a}_i.\vec{a}_j = g_{ij},$$

where the object g_{ij} is not necessarily orthogonal. Hence we get

$$ds^2 = g_{ij} dx_i \, dx_j.$$

This object g_{ij} is a symmetric second rank tensor, it is called the metric tensor.

In the case of cylindrical co-ordinates we saw

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2,$$

so the metric tensor is diagonal. $g_{rr} = 1$, $g_{\theta\theta} = r^2$ and $g_{zz} = 1$.

Generally we write for any orthogonal but curvilinear co-ordinate system:

$$d\vec{s} = \hat{e}_1 h_1 \, dx_1 + \hat{e}_2 h_2 \, dx_2 + \hat{e}_3 h_3 \, dx_3$$

This defines the scale factors h_1, h_2, h_3.

Hence for cylindrical co-ordinates:

$$\begin{aligned} h_r &= 1 \\ h_\theta &= r \\ h_z &= 1 \end{aligned}$$

§**Spherical co-ordinates:**

This is very useful in problems having spherical symmetry, e.g. orbits of planets, Hydrogen atom,...

$$\begin{aligned} x &= r\cos\theta\sin\phi \\ y &= r\sin\theta\sin\phi \\ x &= r\cos\phi \end{aligned}$$

We can easily calculate:

$$\begin{aligned} dx &= dr(\cos\theta\sin\phi) - d\theta\,(r\sin\theta\sin\phi)\ + d\phi(\cos\theta\cos\phi) \\ dy &= dr(\sin\theta\sin\phi) + d\theta\,(r\cos\theta\sin\phi)\ + d\phi(\sin\theta\cos\phi) \\ dz &= dr\cos\phi - r\sin\phi\, d\phi \end{aligned} \tag{1}$$

Substituting into Eq. (4), we find:

$$d\vec{s} = \hat{e}_r dr + \hat{e}_\theta r\sin\theta d\theta + \hat{e}_\phi r d\phi$$

Problem in HW # relates to the details of calculating the unit vectors $\hat{e}_\theta$ etc.

Picture this:

Hence the volume element is

$$dV = dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi,$$

and the square of the arc length

$$ds^2 = dr^2 + r^2\sin^2\theta + r^2 d\phi^2.$$

Hence we can summarize the spherical co-ordinate system by giving the scale factors

$$\begin{aligned} h_r &= 1 \\ h_\theta &= r\sin\theta \\ h_\phi &= r \end{aligned}$$

What can we do with this machinery?

Recall the definition of a gradient:

$$\vec{\nabla} f = \sum \hat{e}_j \frac{\partial f}{\partial s_j}$$

where s_j is the arc length in the jth direction.

$$\vec{\nabla} f = \hat{e}_1 \frac{1}{h_1} \frac{\partial f}{\partial x_1} + \hat{e}_2 \frac{1}{h_2} \frac{\partial f}{\partial x_2} + \hat{e}_3 \frac{1}{h_3} \frac{\partial f}{\partial x_3}.$$

In cylindrical coordinates

$$\vec{\nabla} f = ?$$

In spherical co-ordinates:

$$\vec{\nabla} f = ?$$