Physics 220- Fall 2011

Theory of Many Body Physics

Homework 1

30 September, 2011

1. In taking Fourier series representation on the lattice we often need the following results. By performing the sum explicitly for a finite number of lattice points N, calculate the one dimensional delta type function $D(R_m)$ of the distance $R_m = a_0 m$:

$$D(R_m) = \frac{1}{N} \sum_{n=1,N} e^{iq_n R_m}, \qquad q_n = \frac{2\pi n}{La_0}. \tag{1}$$

With $L = a_0 N$ verify the following:

(1) the periodicity

$$D(R_m) = D(R_m + L),$$

(2) Taking three typical values $N = 4, 16, 64$ plot $D(R_m)$ for $-L/2 \leq R_m \leq L/2$, and thereby convince yourself that to an excellent approximation we may treat

$$D(R_m) = \delta_{m,0}. \tag{2}$$

(3) Verify that the two (d=2) and three (d=3) dimensional generalization of $D(R_m)$ is respectively

$$D(\vec{R}) = \frac{1}{N^d} \sum_{\vec{q}} e^{i\vec{q}.\vec{R}}, \tag{3}$$

where $\vec{R} = a_0\{m_1, m_2, ..m_d\}$ and $\vec{q} = \frac{2\pi}{Na_0}\{n_1, n_2, ..n_d\}$.

2. We are given an Fermionic field operator on the lattice with N sites, satisfying the anticommutation relations (ACR in brief)

$$\{C_n, C_m\} = 0, \quad \{C_n^\dagger, C_m^\dagger\} = 0, \quad \{C_n, C_m^\dagger\} = \delta_{n,m}. \tag{4}$$

We use the Fourier series representation

$$C_n = \frac{1}{\sqrt{N}} \sum_m e^{ik_m n} C(k_m), \quad C_n^\dagger = \frac{1}{\sqrt{N}} \sum_m e^{-ik_m n} C^\dagger(k_m). \tag{5}$$

Show that the relations

$$\{C(k_n), C(k_m)\} = 0, \quad \{C^\dagger(k_n), C^\dagger(k_m)\} = 0, \ \{C(k_n), C^\dagger(k_m)\} = \delta_{n,m}, \tag{6}$$

are true by using the Fourier series results in the first problem. Notice that these relations look invariant under Fourier series, this is a consequence of the unitary nature of Fourier series. More generally given the real space set of Fermi operators show that a unitary transformation:

$$C(\alpha) = \sum_n U(\alpha, n) C_n,$$

with an arbitrary unitary matrix U, that the form of the ACR's are unchanged. These ACR's are known as canonical.

3. Given an operator

$$T = -\sum_{n,1,N} (C_n^\dagger C_{n+1} + C_{n+1}^\dagger C_n),$$

with Fermionic $C's$, calculate this operator in the Fourier basis and show that it is

$$T = -2 \sum_m \cos(k_m a_0) C^\dagger(k_m) C(k_m).$$

This will turn out to be the kinetic energy of Fermions in 1-d, but for now this is an exercise in taking Fourier transforms on the lattice.

4. Work out all equations in Section 3.3 Coleman from (3.24) to (3.42) in detail so that the essential idea of creators and destroyers is firm in your minds.

5. Problem 3 page 50 in Coleman

6. Problem 6 page 51 in Coleman (You can get started on the last two problem but since they is a bit hard, do not expect to complete them right away).