Physics 220- Fall 2011

Theory of Many Body Physics

Examination 1
100 points, Time 1.5 hours
10 October, 2011

1. Consider two Hermitean matrices A and B in D dimensions

$$A_{i,j},\ B_{i,j}\ \ ; 1 \leq i, j \leq D,$$

and two quadratic second quantized operators with Fermi operators

$$\hat{\alpha} = \sum_{i,j} A_{ij}\ C_i^{\dagger} C_j,$$

and

$$\hat{\beta} = \sum_{i,j} B_{ij}\ C_i^{\dagger} C_j,$$

that represent two single particle physical variables (such as the kinetic energy) in a D dimensional quantum system. Express the commutator

$$[\hat{\alpha}, \hat{\beta}],$$

in terms of C_{ij}, the matrix commutator of A and B i.e. $C = [A, B]$. [30]

{ Hint: First write the "master equation" for the commutators of composite operators and then plug in the expressions. }

Solution. The master equation we need is the Fermi case where for arbitrary operators u, v, w we may write

$$[u\ v, w] = \{u, w\}v - u\{w, v\},$$

and by antisymmetry of the commutator

$$[u, v\ w] = \{u, v\}w - v\{w, u\}.$$

Recall that the same equations are valid if we replace antocommutators with commutators. This is the complete set of master commutators.

Let us use these to write

$$
\begin{aligned}
[\hat{\alpha}, \hat{\beta}] &= \sum_{ijkl} A_{ij} B_{kl} [C_i^\dagger C_j, C_k^\dagger C_l] \\
[C_i^\dagger C_j, C_k^\dagger C_l] &= C_i^\dagger [C_j, C_k^\dagger C_l] + [C_i^\dagger, C_k^\dagger C_l] C_j \quad \textit{(using the commutator masters)} \\
&= C_i^\dagger C_l \; \delta_{jk} - C_k^\dagger C_j \delta_{il} \quad \textit{(now using the anti-commutator masters)} \\
\therefore [\hat{\alpha}, \hat{\beta}] &= \sum_{ij} C_{ij} \quad C_i^\dagger C_j.
\end{aligned}
$$

We need to make sure not to confuse C_{ij} a matrix with two indices and the Fermi operators C_j.

2. Consider the kinetic energy operator for a continuum problem of Bosons (spinless)

$$\hat{T} = (-\frac{\hbar^2}{2m}) \int d\mathbf{x} \; \psi^\dagger(\mathbf{x}) \; \nabla^2 \psi(\mathbf{x}).$$

a) Write the number operator $\hat{N}$ in the same representation and show that it commutes with $\hat{T}$. [20]

The number operator is

$$\hat{N} = \int d\mathbf{x} \; \psi^\dagger(\mathbf{x}) \psi(\mathbf{x}).$$

The simplest way to proceed is to use the master relation to establish the two commutators

$$[\psi(\mathbf{x}), \hat{N}] = \psi(\mathbf{x}),$$

and

$$[\psi^\dagger(\mathbf{x}), \hat{N}] = -\psi^\dagger(\mathbf{x}).$$

To give one transparent calculation let me show the first of these

$$
\begin{aligned}
[\psi(\mathbf{x}), \hat{N}] &= \int d\mathbf{x}' \; [\psi(\mathbf{x}), \psi^\dagger(\mathbf{x}')\psi(\mathbf{x}')] \\
&= \int d\mathbf{x}' \; [\psi(\mathbf{x}), \psi^\dagger(\mathbf{x}')]\psi(\mathbf{x}')] + \int d\mathbf{x}' \; \psi^\dagger(\mathbf{x}')[\psi(\mathbf{x}), \psi(\mathbf{x}')] \\
&= \int d\mathbf{x}' \; \delta(\mathbf{x} - \mathbf{x}')\psi(\mathbf{x}') \\
&= \psi(\mathbf{x}).
\end{aligned}
$$

We can now easily show the needed result: (factoring out $(-\frac{\hbar^2}{2m})$)

$$
\begin{aligned}
\int d\mathbf{x} \; \psi^\dagger(\mathbf{x}) \; \nabla_{\mathbf{x}}^2 [\psi(\mathbf{x}), \hat{N}] &= \int d\mathbf{x} \; \psi^\dagger(\mathbf{x}) \; \nabla_{\mathbf{x}}^2 \; [\psi(\mathbf{x}), \hat{N}] + \int d\mathbf{x} \; [\psi^\dagger(\mathbf{x}), \hat{N}] \; \nabla_{\mathbf{x}}^2 \; \psi(\mathbf{x}) \\
&= 0.
\end{aligned}
$$

It is useful to note that the ∇^2 term ignores the operator $\hat{N}$ since the latter is a global operator (i.e. integrates over all space).

b) Calculate the commutator

$$[\hat{T}, \psi^\dagger(\mathbf{x})].$$

[20]

Again dropping the factor $(-\frac{\hbar^2}{2m})$, we consider

$$\begin{aligned}
\int d\mathbf{x}' \; [\psi^\dagger(\mathbf{x}') \; \nabla^2_{\mathbf{x}'} \; \psi(\mathbf{x}'), \psi^\dagger(\mathbf{x})] &= \int d\mathbf{x}' \; \psi^\dagger(\mathbf{x}') \; \nabla^2_{\mathbf{x}'} \; [\psi(\mathbf{x}'), \psi^\dagger(\mathbf{x})] \\
&= \int d\mathbf{x}' \; \psi^\dagger(\mathbf{x}') \; \nabla^2_{\mathbf{x}'} \; \delta(\mathbf{x} - \mathbf{x}') \\
&= \int d\mathbf{x}' \; (\nabla^2_{\mathbf{x}'} \psi^\dagger(\mathbf{x}')) \;\; \delta(\mathbf{x} - \mathbf{x}') \;\; \textit{(integrating by parts)} \\
&= \nabla^2_{\mathbf{x}} \psi^\dagger(\mathbf{x}) \\
\therefore [\hat{T}, \psi^\dagger(\mathbf{x})] &= (-\frac{\hbar^2}{2m}) \nabla^2_{\mathbf{x}} \psi^\dagger(\mathbf{x})
\end{aligned}$$

3. Consider the field operator for a spinless Boson

$$\psi^\dagger(\mathbf{x}) = \sum_\gamma \phi^*_\gamma(\mathbf{x}) \; b^\dagger_\gamma,$$

where b_α is a Bosonic mode operator and $\phi_\gamma(\mathbf{x})$ is a coordinate space wave function. Express the two Boson state

$$|a\rangle = b^\dagger_\alpha \; b^\dagger_\beta |0\rangle,$$

as a wave function for two particles in the usual coordinate representation by taking the overlap

$$\langle 0|\psi(\mathbf{x}_1)\psi(\mathbf{x}_2)|a\rangle.$$

This can be done as follows

$$\begin{aligned}
\langle 0|\psi(\mathbf{x}_1)\psi(\mathbf{x}_2)|a\rangle &= \sum_{\mu\nu} \phi_\mu(\mathbf{x}_1)\phi_\nu(\mathbf{x}_2) \; \langle 0|b_\mu b_\nu b^\dagger_\alpha b^\dagger_\beta|0\rangle \\
&= \sum_{\mu\nu} \phi_\mu(\mathbf{x}_1)\phi_\nu(\mathbf{x}_2)(\delta_{\mu\alpha}\delta_{\nu\beta} + \delta_{\nu\alpha}\delta_{\mu\beta}) \\
&= (\phi_\alpha(\mathbf{x}_1)\phi_\beta(\mathbf{x}_2) + \phi_\beta(\mathbf{x}_1)\phi_\alpha(\mathbf{x}_2)).
\end{aligned}$$

The expectation of the Bose operators is done as in the Harmonic oscillator problem, by pushing the destruction ops to the right. [30]